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THE TOPOLOGY OF GENERALIZED FUZZY METRIC SPACES AND VECTOR IMAGE FILTERING

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Abstract: Khan generalized the concept of Fuzzy metric space (In the sense of George and Veeramani) and introduced the notion of Generalized fuzzy n-metric spaces. In this paper, we further investigate the properties of these generalized fuzzy metric spaces and extend the Banach Fixed Point Theorem in this new framework. We also propose a vector image filter based on generalized fuzzy metrics.

1. INTRODUCTION

Kramosil and Michalek [1] introduced the notion of a fuzzy metric space by generalizing the concept of a probabilistic metric space introduced by K. Menger. George and Veer Amani [2], used the concept of continuous t-norms to modify this definition of fuzzy metric space and showed that the topology for their new definition is Hausdorff () . These fuzzy spaces have important applications in image filtering and quantum particle physics. In 1988, Grabiec [3] first defined the Banach contraction in a fuzzy metric space and extended fixed point theorems of Banach and Edelstein to fuzzy metric spaces. Following this approach, Vasuki [4] generalized Grabiec's fuzzy Banach contraction theorem and proved a common fixed-point theorem for a sequence of mappings in a fuzzy metric space Tanaka, Mizuno and Kado, 2005 [5]. Sharma [6] also extended some known results of fixed-point theory for compatible mappings in fuzzy metric spaces. In 2002, Gregori and Sapena [7] introduced the notion of fuzzy contractive mapping and proved some fixed-point theorems for complete fuzzy metric spaces in the sense of George and Veer Amani. Mihet [8] and Sedghi and Shobe, 2006 [9] proposed a fuzzy Banach theorem for (weak) B-contraction in M-complete fuzzy metric spaces. There are several generalizations of fuzzy metric spaces for more than two variables. Recently Khan generalized and studied the concept of Generalized fuzzy n-metric space by combining the definition given by George and Veer Amani [10], with that of generalized n-metric space. In this paper, we further investigate the properties of generalized fuzzy n-metric spaces and prove the famous Banach fixed point theorem in this new framework Schweizer *et al.*, 1960 [11]. Fuzzy metrics are found to be useful in color image filtering techniques Gregori, Miñana and Morillas, 2012 [12]. Morillas et al [13] replaced the classical metrics by fuzzy metrics (in the sense of George and Veeramani) in constructing a variant of Vector Median Filter (VMF) . Recently Khan [14] proposed a more generalized version of these filters using generalized fuzzy n-metric spaces. In this paper, we propose a vector filter based on Khan's proposal for color image processing.

Definition 1. A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

- $*$ is associative and commutative;
- $*$ is continuous;
- $a * 1 = a$ for all $a \in [0,1]$;
- $a * b \le c * d$ whenever $a \le c$ and $b \le d$

for each $a, b, c, d \in [0,1]$. The examples of continuous t-norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 2. (George and Veeramani) A Fuzzy Metric Space is a triple $(X, M, *)$ where X is a nonempty set, $*$ is a continuous t-norm and $M: X \times X \times (0, \infty) \to [0,1]$ is a mapping (called fuzzy metric) which satisfies the following properties: for every $x, y, z \in X$ and $s, t > 0$

-] $M(x, y, t) > 0$;
-] $M(x, y, t) = 1$ if and only if $x = y$;
-] $M(x, y, t) = M(y, x, t)$;
-] $M(x, z, t + s) \ge M(x, y, t) * M(y, z, s)$
-] $M(x, y, *): (0, \infty) \to (0,1]$ is continuous.

Then M is called a fuzzy metric on X and $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

From now on, by a fuzzy metric $M(x, y, t)$We always means a fuzzy metric in the sense of George and Veeramani.

Example 3. Let X be a non-empty set and d is a metric on X. Denote $a * b = a.b$ for all $a, b \in [0,1]$. For each $t > 0$, define

$$M_d(x,y,t)=\frac{t}{t+d(x,y)}$$

Then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric M_d induced by the metric d the standard fuzzy metric.

We call a topological space (X,τ) fuzzy metrizable if there exists a fuzzy metric M on X such that $\tau=\tau_M$. George and Veeraman, 1994, showed that every fuzzy metric M on X generates a topology τ_M on X. The family of open sets $\{B_M(x,r,t): x\in X,\ 0<r<1,\ t>0\}$ forms a base for this topology Khan, 2012 [15], where $B_M(x,r,t)=\{y\in X: M(x,y,t)>1-r\}$ for every $r, 0<r<1$ and $t>0$. They also proved that this topological space is first countable and Hausdorff. Also, for a metric space (X,d), the topology generated by d coincides with the topology τ_{M_d} generated by the standard fuzzy metric M_d thereby indicating that every metrizable topological space is fuzzy metrizable. Gregory and Romaguera [16], proved that the collection $\{U_n: n\in\mathbb{N}\}$ is a base for uniformity $\mathcal{U}_M$ compatible with τ_M, where $U_n=\{(x,y): M\left(x,y,\frac{1}{n}\right)>1-1/n\}$ for all $n\in\mathbb{N}$. Hence the topological space (X,τ_M)S metrizable.

Definition 4. A 3-tuple $(X, F_n, *)$ is called Generalized Fuzzy n-metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and F_n is a fuzzy set on $X^n\times(0,\infty)$ satisfying the following conditions for each $x_1,x_2,\dots x_n\in X$ and $t,s>0$:

-] $F_n(x_1,x_1,\dots,x_1,x_2,t)>0$ for all $x_1,x_2\in X$ with $x_1\neq x_2$;
-] $F_n(x_1,x_1,\dots,x_2,t)\geq F_n(x_1,x_2,\dots,x_n,t)$ with the condition that at least two of the points $x_2,x_3,\dots x_n$ are distinct;
-] $F_n(x_1,x_2,\dots,x_n,t)=1$ if and only if $x_1=x_2=\dots=x_n$;
-] $F_n(x_1,x_2,\dots,x_n,t)=F_n\left(x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(n)},t\right)$ for every permutation π of $\{1,2,\dots,n\}$;
-] $F_n(x_1,x_{n+1},\dots,x_{n+1},t)*F_n(x_{n+1},x_2,\dots,x_n,s)\leq F_n(x_1,x_2,\dots,x_n,t+s)$;
-] $F_n(x_1,x_2,\dots,x_n,.):(0,\infty)\to[0,1]$ is continuous.

Example 5. Let (X,G_n) be a Generalized n-metric space. Denote $a*b=a.b$ for all $a,b\in[0,1]$. For each $t>0$, define

$$F_n(x_1,x_2,\dots,x_n,t)=\frac{t}{t+G_n(x_1,x_2,\dots,x_n)} \quad (1)$$

for all $x_1,x_2,\dots x_n\in\mathbb{R}$. Then $(X,F_n,*)$ is a Generalized fuzzy n-metric space.

Proposition 6. Let $(X,F_n,*)$ be a generalized fuzzy n-metric space. Then for $x,y\in X$ and $t>0$, we have $F_n(x,y,y,\dots y,t)\geq [F_n\left(y,x,x,..x,\frac{t}{n-1}\right)]^{n-1}$

Definition 7. Let $(X,F_n,*)$ be a generalized fuzzy n-metric space. A subset A of X is said to be F-bounded if there exists $t>0$ and $r\in(0,1)$ such that

$$F_n(x_1,x_2,\dots,x_n,t)>1-r \text{ for all } x_1,x_2,\dots,x_n\in A \quad (2)$$

Definition 8. A generalized fuzzy n-metric $(F_n,*)$ on X is said to be stationary if F_n does not depend on t, i.e. for each $x_1,x_2,\dots,x_n\in X$ the function $F_n(x_1,x_2,\dots,x_n,t)$ is constant.

Proposition 9. *Let X be a closed real interval $[a,b]$ and $K>|a|>0$. Consider for each $n=1,2,\dots$ the function $F_r{}^{(n)}: \underbrace{X^n\times X^n\times\dots\times X^n}_{r}\times(0,\infty)\to(0,1]$ given by $F_r{}^{(n)}(x^1,x^2,\dots,x^r,t)=\prod_{i=1}^{n}\frac{\min\{x_i^1,x_i^2,\dots,x_i^r\}+K}{\max\{x_i^1,x_i^2,\dots,x_i^r\}+K}$ Where $x^j=\left(x_1^j,x_2^j,\dots,x_n^j\right)$, $j=1,2,\dots,r$ and $t>0$. Then $\left(X^n,F_r{}^{(n)},*\right)$ is a stationary F-bounded Generalized fuzzy r-metric space, where $a*b=a\cdot b$ for all $a,b\in[0,1]$.*

Definition 10. Let $(X,F_n,*)$ be a Generalized fuzzy n-metric space. For $t>0$, the open ball $B_F(x_0,r,t)$ with center x_0 and radius $0<r<1$ is defined by

$$B_F(x_0,r,t)=\{y\in X: F_n(x_0,y,y,\dots,y,t)>1-r\} \quad (3)$$

Definition 11. A subset A of X is called an open set if for each $x\in A$ there exist $t>0$ and $0<r<1$ such that $B_F(x,r,t)\subset A$.

2. MAIN RESULTS

Proposition 12. Let $(X,F_n,*)$ be a generalized fuzzy n-metric space.

If $F_n(x_1,x_2,\dots,x_n,T)>1-r$ for $x_1,x_2,\dots,x_n\in X$, $T>0$, $0<r<1$, We can find a t_0, $0<t_0<T$ such that $F_n(x_1,x_2,\dots,x_n,t_0)>1-r$.

Proof. Let $\tilde{t}=\text{Sup}\{t>0: F_n(x_1,x_2,\dots,x_n,t)\leq 1-r\}$, then $\tilde{t}>0$. Since $F_n(x_1,x_2,\dots,x_n,t)$ is nondecreasing with respect to t, hence

$$F_n(x_1, x_2, \ldots, x_n, t) > 1 - r \text{ for all } t > \tilde{t} \quad (4)$$

Therefore $T > \tilde{t}$ and there exists $t_0 > 0$ such that $\tilde{t} < t_0 < T$ and \$F_n(x_1,x_2,...,x_n,t_0)\\>1-r\$. □

Proposition 13. Every open ball in a Generalized fuzzy n-metric space is an open set.

Proof. Let $(X, F_n, *)$ be a generalized fuzzy n-metric space. Consider an 0open ball $B_F(x, r, t)$. Let $y \in B_F(x, r, t)$, hence we have $F_n(x, y, \ldots, y, t) > 1 - r$. We shall show that there exists $R > 0$ and $T > 0$ such that

$$B_F(y, R, T) \subset B_F(x, r, t) \quad (5)$$

By **Proposition 12,** there exists t_0, $0 < t_0 < t$, such that $F_n(x, y, \ldots, y, t_0) > 1 - r$. Let $F_n(x, y, \ldots, y, t_0) = r_1$. Then there exists s, $0 < s < 1$ such that $r_1 > 1 - s > 1 - r$. Now for a given r_1 and sWe can find r_2, $0 < r_2 < 1$, such that

$$r_1 * r_2 \geq 1 - s \quad (6)$$

Let $R = 1 - r_2$ and $T = t - t_0$. Then $z \in B_F(y, R, T)$ implies that$F_n(y, z, \ldots, z, t - t_0) > r_2$. Thus

$$\begin{aligned} F_n(x, z, \ldots, z, t) &\geq F_n(x, y, \ldots, y, t_0) * F_n(y, z, \ldots, z, t - t_0) \\ &> r_1 * r_2 \\ &\geq 1 - s \\ &> 1 - r \end{aligned} \quad (7)$$

Thus $z \in B_F(x, r, t)$ and hence $B_F(y, R, T) \subset B_F(x, r, t)$. □

Proposition 14. Let $(X, F_n, *)$ be a generalized fuzzy n-metric space. Let $B_F(x, r_1, t)$ and $B_F(x, r_2, t)$ be open balls with the same center $x \in X$ and with radius $0 < r_1 < 1$ and $0 < r_2 < 1$ respectively. Then either we have $B_F(x, r_1, t) \subseteq B_F(x, r_2, t)$ or $B_F(x, r_2, t) \subseteq B_F(x, r_1, t)$.

Proof. For $r_1 = r_2$, The result is obvious. Suppose that $r_1 \neq r_2$. Now there are two possibilities viz. $0 < r_1 < r_2 < 1$ and $0 < r_2 < r_1 < 1$. For the case $0 < r_1 < r_2 < 1$, We have $1 - r_2 < 1 - r_1$. Let $y \in B_F(x, r_1, t)$, then

$$F_n(x, y, \ldots, y, t) > 1 - r_1 > 1 - r_2 \quad (8)$$

Hence $y \in B_F(x, r_2, t)$ implying that $B_F(x, r_1, t) \subseteq B_F(x, r_2, t)$. A similar argument can be given for the case $0 < r_2 < r_1 < 1$. □

Proposition 15. Let $(X, F_m, *)$ be a generalized fuzzy m-metric space. Then (X, τ_F) is first countable.

Proof. Let $t > 0$ and $x \in X$. We will show that

$$H_x = \{B_F\left(x, \frac{1}{n}, \frac{t}{n}\right) : n \in \mathbb{N}\} \quad (9)$$

is a local base for $x \in X$.

Let $U \in \tau_F$ and $x \in U$. Since U is open, there exists $0 < r < 1$, $t > 0$ such that

$$B_F(x, r, t) \subset U \quad (10)$$

Choose $n \in \mathbb{N}$ such that $r > \frac{1}{n}$. Let $z \in B_F\left(x, \frac{1}{n}, \frac{t}{n}\right)$, then

$$\begin{aligned} & F_m\left(x, z, z, \ldots, z, \frac{t}{n}\right) > 1 - \frac{1}{n} > 1 - r \\ \Longrightarrow\quad & F_m(x, z, z, \ldots, z, t) \geq F_m\left(x, z, z, \ldots, z, \frac{t}{n}\right) > 1 - r \\ \Longrightarrow\quad & z \in B_F(x, r, t) \end{aligned} \quad (11)$$

Therefore $B_F\left(x, \frac{1}{n}, \frac{t}{n}\right) \subset B_F(x, r, t) \subset U$. Hence H_x is a countable local base for x, i.e. (X, τ_F)It is a first-countable topological space. □

Definition 16. Let $(X, F_n, *)$ be Generalized fuzzy n-metric space and $A \subseteq X$. Let Λ be an index set. A collection $\{G_\alpha : \alpha \in \Lambda\}$ of open sets in X is called an open cover of A if $A \subseteq \{G_\alpha : \alpha \in \Lambda\}$.

Definition 17. A subset A of a Generalized fuzzy n-metric space $(X, F_n, *)$ is said to be compact if every open cover $\tilde{G}$ of A has a finite subcover.

Proposition 18. Every compact subset A of a generalized fuzzy n-metric space X is F-bounded.

Proof. Let $(X, F_n, *)$ be Generalized fuzzy n-metric space and A be a compact subset of X. For fixed values of $t > 0$ and $0 < r < 1$, the collection $\{B_F(x, r, t): x \in A\}$ is an open cover of A. Since A is compact, there exists $x_1, x_2, \dots, x_k \in A$ such that

$$A \subseteq \bigcup_{i=1}^{k} B_F(x_i, r, t) \tag{12}$$

Let $z_1, z_2, \dots, z_n \in A$. Then there exists a subset $\{x_{i_1}, x_{i_2}, \dots, x_{i_n}\}$ of $\{x_i: i = 1,2, \dots, k\}$, where x_{i_p}sets are not necessarily distinct, such that $z_p \in B_F\left(x_{i_p}, r, t\right)$. Hence, we have

$$F_n\left(x_{i_p}, z_p, \dots, z_p, t\right) > 1 - r \tag{13}$$

Therefore, on using proposition 6,we have

$$F_n\left(z_p, x_{i_p}, \dots, x_{i_p}, (n-1)t\right) \geq [F_n\left(x_{i_p}, z_p, \dots, z_p, t\right)]^{n-1} > (1 - r)^{n-1} \tag{14}$$

Now using [M 5], we have

$$\begin{aligned} F_n(z_1, z_2, \dots, z_n, (n^2 - 1)t) \quad & \geq F_n\big(z_1, x_{i_1}, \dots, x_{i_1}, (n-1)t\big) * F_n\big(z_2, x_{i_2}, \dots, x_{i_2}, (n-1)t\big) * \dots \\ & \dots * F_n\big(z_n, x_{i_n}, \dots, x_{i_n}, (n-1)t\big) * F_n\big(x_{i_1}, x_{i_2}, \dots, x_{i_n}, (n-1)t\big) \end{aligned} \tag{15}$$

Let $r_1 = \min\{F_n\big(x_{i_1}, x_{i_2}, \dots, x_{i_n}, t\big): 1 \leq i_1, i_2, \dots, i_n \leq k\}$, Then we have

$$F_n\big(x_{i_1}, x_{i_2}, \dots, x_{i_n}, (n-1)t\big) \geq F_n\big(x_{i_1}, x_{i_2}, \dots, x_{i_n}, t\big) \geq r_1 \tag{16}$$

In view of relations [fuzzyeq2], [fuzcond4] and [fuzzyeq3], We have

$$F_n(z_1, z_2, \dots, z_n, (n^2 - 1)t) > \underbrace{(1 - r)^{n-1} * (1 - r)^{n-1} * \dots * (1 - r)^{n-1}}_{n} * r_1 \tag{17}$$

Let $(n^2 - 1)t = t_0$ and

$$\underbrace{(1 - r)^{n-1} * (1 - r)^{n-1} * \dots * (1 - r)^{n-1}}_{n} * r_1 > 1 - s,\ 0 < s < 1 \tag{18}$$

We have

$$F_n(z_1, z_2, \dots, z_n, t_0) > 1 - s \tag{19}$$

for all $z_1, z_2, \dots, z_n \in A$. Hence A is F-bounded. □

Definition 19. Let $(X, F_k, *)$ be Generalized fuzzy k-metric space.Let $< ((u_1)_n, (u_2)_n \dots (u_k)_n, t_n) >$ be a sequence in $X^k \times (0, \infty)$ converging to a point $(x_1, x_2, \dots, x_k, t)$ in $X^k \times (0, \infty)$. Then F_k is said to be continuous on $X^k \times (0, \infty)$ if

$\lim_{n\to\infty} F_k((u_1)_n, (u_2)_n \dots (u_k)_n, t_n) = F_k(x_1, x_2, \dots x_k, t)$ (17)

Proposition 20. Let $(X, F_k, *)$ be Generalized fuzzy k-metric space. Then F_k is continuous on $X^k \times (0, \infty)$.

Proof. Let $x_1, x_2, \dots, x_k \in X$ and $t > 0$. Consider a sequence $< ((v_1)_n, (v_2)_n \dots (v_k)_n, s_n) >$ in $X^k \times (0, \infty)$ converging to $(x_1, x_2, \dots, x_k, t)$. Then $< F_k((v_1)_n, (v_2)_n \dots (v_k)_n, s_n) >$ is a sequence in $(0,1]$. Hence there exists a subsequence $< ((u_1)_n, (u_2)_n \dots (u_k)_n, t_n) >$ of the sequence $< ((v_1)_n, (v_2)_n \dots (v_k)_n, s_n) >$ such that $< F_k((u_1)_n, (u_2)_n \dots (u_k)_n, t_n) >$ converges to some point in $[0,1]$. Now $t_n \to t$ as $n \to \infty$, hence for given $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $|t_n - t| < \delta$ for all $n \geq n_0$. Let us choose $\delta < \frac{t}{2}$. Then for all $n \geq n_0$, we have

$$\begin{aligned} F_k((u_1)_n, (u_2)_n \dots (u_k)_n, t_n) \quad & \geq F_k((u_1)_n, (u_2)_n \dots (u_k)_n, t - \delta) \\ & \geq F_k\left((u_1)_n, x_1 \dots x_1, \tfrac{\delta}{k}\right) * F_k\left(\left(x_1, (u_2)_n \dots (u_k)_n, t - \tfrac{(k+1)\delta}{k}\right)\right. \\ & \quad \vdots \\ & \geq F_k\left((u_1)_n, x_1 \dots x_1, \tfrac{\delta}{k}\right) * F_k\left((u_2)_n, x_2 \dots x_2, \tfrac{\delta}{k}\right) * \dots \\ & \quad \dots F_k\left((u_k)_n, x_k \dots x_k, \tfrac{\delta}{k}\right) * F_k((x_1, x_2 \dots x_k, t - 2\delta) \end{aligned} \tag{20}$$

and

$$\begin{aligned} F_k(x_1,x_2,\dots x_k,t+2\delta) &\geq F_k(x_1,x_2,\dots x_k,t_n+\delta) \\ &\geq F_k\left(x_1,(u_1)_n,\dots(u_1)_n,\frac{\delta}{k}\right) * F_k\left(x_2,(u_2)_n,\dots(u_2)_n,\frac{\delta}{k}\right) * \dots \\ &\quad \dots * F_k\left(x_k,(u_k)_n,\dots(u_k)_n,\frac{\delta}{k}\right) * F_k((u_1)_n,(u_2)_n,\dots(u_k)_n,t_n) \end{aligned} \tag{21}$$

Making $n \to \infty$ we obtain

$$\lim_{n\to\infty} F_k((u_1)_n,(u_2)_n \dots (u_k)_n,t_n) \geq 1 * 1 * \dots * 1 * F_k(x_1,x_2 \dots x_k,t-2\delta) = F_k((x_1,x_2 \dots x_k,t-2\delta) \tag{22}$$

and

$F_k(x_1,x_2,\dots x_k,t+2\delta) \geq \lim_{n\to\infty} F_k((u_1)_n,(u_2)_n,\dots(u_k)_n,t_n)$ (21)

respectively. Therefore

$$F_k((x_1,x_2 \dots x_k,t-2\delta) \leq \lim_{n\to\infty} F_k((u_1)_n,(u_2)_n,\dots(u_k)_n,t_n) \leq F_k(x_1,x_2,\dots x_k,t+2\delta) \tag{23}$$

Since $F_k(x_1,x_2,\dots,x_k,.):(0,\infty) \to [0,1]$ is continuous, hence we have

$$\lim_{n\to\infty} F_k((u_1)_n,(u_2)_n,\dots(u_k)_n,t_n) = F_k(x_1,x_2,\dots x_k,t) \tag{24}$$

Therefore F_k is continuous on $X^k \times (0,\infty)$. □

Now we extend the Fuzzy Banach contraction theorem in the framework of Generalized fuzzy n-metric space.

Theorem 21. Let $(X,F_r,*)$ be a F_r-complete Generalized fuzzy r-metric space. Let $T:X\to X$ be a mapping satisfying $F_r(Tx_1,Tx_2,\dots,Tx_r,kt) \geq F_r(x_1,x_2,\dots x_r,t)$ for all $x_1,x_2,\dots,x_r \in X$ and $0<k<1$. Then T has a unique fixed point.

Proof. Let $y_0 \in X$. Consider a sequence $< y_n >$ in X such that $y_n = T^n y_0$. By Condition [fuzcond1], we have

$$\begin{aligned} F_r(Ty_{n-1},Ty_n,\dots,Ty_n,kt) &\geq F_r(y_{n-1},y_n,\dots,y_n,t) \\ \Rightarrow F_r(y_n,y_{n+1},\dots,y_{n+1},kt) &\geq F_r(y_{n-1},y_n,\dots,y_n,t) \\ \Rightarrow F_r(y_n,y_{n+1},\dots,y_{n+1},t) &\geq F_r\left(y_{n-1},y_n,\dots,y_n,\frac{t}{k}\right) \\ &\geq F_r\left(y_{n-2},y_{n-1},\dots,y_{n-1},\frac{t}{k^2}\right) \\ &\vdots \\ &\geq F_r\left(y_0,y_1,\dots,y_1,\frac{t}{k^n}\right) \end{aligned} \tag{25}$$

We claim that the sequence $< y_n >$ is a F_r-Cauchy sequence in X.
For all natural numbers n and p We have

$$\begin{aligned} F_r(y_n,y_{n+p},\dots,y_{n+p},t) &\geq F_r\left(y_n,y_{n+1},\dots,y_{n+1},\frac{t}{p-1}\right) * F_r\left(y_{n+1},y_{n+p},\dots,y_{n+p},\frac{t(p-2)}{p-1}\right) \\ &\geq F_r\left(y_n,y_{n+1},\dots,y_{n+1},\frac{t}{p-1}\right) * F_r\left(y_{n+1},y_{n+2},\dots,y_{n+2},\frac{t}{p-1}\right) \\ &\quad * F_r\left(y_{n+2},y_{n+p},\dots,y_{n+p},\frac{t(p-2)}{p-1}\right) \\ &\quad \vdots \\ &\geq F_r\left(y_n,y_{n+1},\dots,y_{n+1},\frac{t}{p-1}\right) * F_r\left(y_{n+1},y_{n+2},\dots,y_{n+2},\frac{t}{p-1}\right) \\ &\quad \dots * F_r\left(y_{n+p-1},y_{n+p},\dots,y_{n+p},\frac{t}{p-1}\right) \\ &\geq F_r\left(y_0,y_1,\dots,y_1,\frac{t}{k^n(p-1)}\right) * F_r\left(y_0,y_1,\dots,y_1,\frac{t}{k^{n+1}(p-1)}\right) \\ &\quad \dots * F_r\left(y_0,y_1,\dots,y_1,\frac{t}{k^{n+p-1}(p-1)}\right) \end{aligned} \tag{26}$$

Since the t-norm $*$ is continuous and $F_r(x_1,x_2,\dots,x_n,.)$ is continuous, hence making $n\to\infty$, we have

$$\lim_{n\to\infty} F_r(y_n,y_{n+p},\dots,y_{n+p},t) \geq 1 * 1 \dots * 1 = 1 \tag{27}$$

This shows that $< y_n >$ is a F_r-Cauchy sequence. Since X is F_r-Complete, there exists a point $u \in X$ such that $< y_n >$ is F_r-convergent and converges to u . suppose $Tu \neq u$ then

$$\begin{aligned} F_r(Tu,u,\dots,u,t) &\geq F_r\left(Tu,Ty_n,\dots,Ty_n,\frac{t}{2}\right) * F_r\left(Ty_n,u,\dots,u,\frac{t}{2}\right) \\ &\geq F_r\left(u,y_n,\dots,y_n,\frac{t}{2k}\right) * F_r\left(y_{n+1},u,\dots,u,\frac{t}{2}\right) \end{aligned} \tag{28}$$

Taking the limit as $n \to \infty$ and using the fact that function F_r is continuous on its variables, we have

$$F_r(Tu,u,\dots,u,t) \to 1 \tag{29}$$

This implies $Tu = u$.
Next to show the uniqueness, suppose there exists $v \in X$ such that $Tv = v$. Then

$$\begin{aligned} F_r(v,u,\dots,u,t) &= F_r(Tv,Tu,\dots,Tu,t) \\ &\geq F_r\left(v,u,\dots u,\frac{t}{k}\right) \\ &= F_r\left(Tv,Tu,\dots,Tu,\frac{t}{k}\right) \\ &\geq F_r\left(v,u,\dots,u,\frac{t}{k^2}\right) \\ &\vdots \\ &\geq F_r\left(v,u,\dots,u,\frac{t}{k^n}\right) \to 1 \quad \text{as} \quad n \to \infty \end{aligned} \tag{30}$$

Which implies $v = u$. □

We now give an example to illustrate Theorem 3.14.

Example 22. Let $X = [-1,1]$, $a * b = a.b$ for all $a,b \in [0,1]$ and $t > 0$, define

$$F_n(x_1,x_2,\dots,x_n,t) = \frac{t}{t+\sum_{i,j=1,i<j}^{n}|x_i-x_j|} \tag{31}$$

for all $x_1,x_2,\dots x_n \in X$. Clearly $(X,F_n,*)$ is a F_n-complete Generalized fuzzy n-metric space. Define a mapping $T:X \to X$ by $Tx = x/6$ for all $x \in X$. One can see that the condition [fuzcond1] holds for all $x_1,x_2,\dots x_n \in X$ and $\frac{1}{6} \leq k < 1$ and 0 is the unique fixed point of T.

3. APPLICATION TO VECTOR IMAGE FILTERING

The quality of digital images is affected by the sensor noise and the channel noise. The sensor noise is produced during image formation while the channel noise is produced during transmission by Plataniotis [17]. Therefore, it is essential to reduce the noise for estimating the original image information from noisy data Plataniotis, Androutsos [18]. This process is called the color image filtering and is an important part of any color image processing system Machuca and Phillips, 1983 [19]. There are several approaches to constructing a color filter for this purpose. The vector approach Astola, Haavisto and Neuvo, 1990 [20] is observed to be more appropriate compared to other traditional approaches. The most well-known filter following this approach is a Vector Median Filter (VMF) provided by Arakawa, 1996 [21]. In this approach, the color images are treated as a vector field, and a window is moved over the input image. The vector filter selects the output vector on the basis of ordering of vectors in the defined sliding window.

Let W be the sliding window of size n and let $x_i, i = 1,2,\dots,n$ be the pixels in W. Let the vector-valued image function at pixel x_j be denoted by $I_j = \left(I_j(1),I_j(2),I_j(3)\right)$. For RGB images, $I_j \in \{0,1,\dots,255\}^3$. The degree of nearness or closeness between two vectors I_i and I_j is described by

$$d\left(I_i,I_j\right) = \| I_i - I_j \| = (\textstyle\sum_{l=1}^{3}\left|I_i(l) - I_j(l)\right|^p)^{1/p} \tag{32}$$

Let the aggregated distance associated with the input vector I_i is given by

$$D^i = \textstyle\sum_{j=1}^{n} d\left(I_i,I_j\right) \tag{33}$$

Then the order sequence of aggregated distances $D^{(1)} \leq D^{(2)} \leq \cdots \leq D^{(n)}$ implies the same ordering of corresponding vectors $I_{(1)} \leq I_{(2)} \leq \cdots I_{(n)}$. The VMF outputs the vector $\widetilde{I_k}$ that minimizes the aggregated distances to the other vectors in W, i.e. $I_{\text{out}} = \widetilde{I_k}$ for which

$$k = \underset{i}{\text{argmin}}(D^i), \quad i = 1,2,\dots,n \tag{34}$$

Hence the lowest-ranked vector $I_{(1)}$ is the output of VMF.

It has been observed that fuzzy-based methods are useful in the detection and removal of noise during image processing [22] . Morillas et al and Pitas [23] replaced the classical metric defining nearness between pixels I_i and I_j by the following fuzzy metric-

$$M(I_i, I_j) = \prod_{l=1}^{3} \frac{\min\{I_i(l), I_j(l)\}+K}{\max\{I_i(l), I_j(l)\}+K} \tag{35}$$

Where $I_i(l), I_j(l) \in$ integer values in $[0,255]$ for the processing of RGB images and $K > 0$. Recently Khan, Sun and Yang, [24] proposed to replace the fuzzy metric $M(I_i, I_j)$ by generalized fuzzy metric $F_r^{(3)}(I_1, I_2, \ldots, I_r)$ defined in proposition 9. Then the accumulated fuzzy measure associated to the vector I_i is given by

$$D^i = \sum_{\substack{1\le i_1<i_2<\ldots<i_{r-1}\le n \\ i_k\ne i,\, k=1,2,\ldots,r-1}} F_r^{(3)}\left(I_i, I_{i_1}, \ldots, I_{i_{r-1}}\right) \tag{36}$$

For different values of r, We have different algorithms reducing the noise in color image processing, Machuca and Phillips, 1983 [19].

Now we propose an algorithm by choosing $r = 4$ for a 8-neighbourhood 3×3 window. For this choice of $r(= 4)$, We have a new vector filter Xu and Yue, 2009 [25].

with accumulated fuzzy measure associated to the vector I_i as

$$D^i = \sum_{\substack{1\le i_1<i_2<i_3\le 9 \\ i_k\ne i,\, k=1,2,3}} F_4^{(3)}\left(I_i, I_{i_1}, I_{i_2}, I_{i_3}\right) \tag{37}$$

Where $I_{i_p} = \left(I_{i_p}(1), I_{i_p}(2), I_{i_p}(3)\right)$, $I_{i_p}(l) \in \{0,1,2,\ldots,255\}$ for $l = 1,2,3;$ $p = 1,2,3$, and

$$F_4^{(3)}(I_i, I_{i_1}, I_{i_2}, I_{i_3}) = \prod_{l=1}^{3} \frac{\min\{I_i(l), I_{i_1}(l), I_{i_2}(l), I_{i_3}(l)\}+K}{\max\{I_i(l), I_{i_1}(l), I_{i_2}(l), I_{i_3}(l)\}+K} \tag{38}$$

for the processing of RGB images and $K > 0$. The appropriate value of K can be decided by analyzing the performance (MSE) of the fuzzy metric $F_4^{(3)}$ with respect to different values of K. The fuzzy metric $F_4^{(3)}$ is a particular stationary form of the generalized metric $F_4(x, y, z, w, t)$. An interpretation of such metric can be given as follows-

$F_4(x, y, z, w, t) = \alpha$ if and only if the probability $P[(\text{Perimeter of the rectangle with} \; \text{vertices} \; x,y,z \; \text{and} \; w)\le t]=\alpha$.

Then the accumulated fuzzy measure D^i associated with the vector I_i will represent the sum of fuzzy analog perimeters of all the rectangles formed with the pixel I_i as one vertex.

Let us consider a 8-neighbourhood 3×3 window with pixels $\{(i,j): i,j \in \{1,2,3\}\}$. If we number the pixel (i,j) as $(3i + j - 3)$, Then the pixels numbered 1,3,7 and 9 will have similar spatial neighborhoods. The pixels numbered 2,4,6 and 8 are also spatially similar with respect to the sliding window. The pixel numbered 5 is the central pixel. Now We propose the algorithm for computing D^i as follows-

$$\begin{aligned} D^1 &= F_4^{(3)}(I_1, I_2, I_4, I_5) + F_4^{(3)}(I_1, I_3, I_7, I_9) + F_4^{(3)}(I_1, I_1, I_6, I_8), \\ D^2 &= F_4^{(3)}(I_2, I_1, I_3, I_5) + F_4^{(3)}(I_2, I_4, I_6, I_8) + F_4^{(3)}(I_2, I_2, I_7, I_9) \\ D^5 &= F_4^{(3)}(I_5, I_1, I_2, I_4) + F_4^{(3)}(I_5, I_6, I_8, I_9) + F_4^{(3)}(I_5, I_3, I_5, I_7) \end{aligned} \tag{39}$$

We can follow a similar approach to compute D^i for other spatially similar pixels numbered 3,7,9 and 4,6,8.

The filter output will be $\widetilde{I_k} \in W$ that maximizes the aggregated fuzzy measure to other vectors in W, i.e. $I_{\text{out}} = \widetilde{I_k}$ for which

$$k = \operatorname*{argmax}_i (D^i), \quad i = 1,2,\ldots,9 \tag{40}$$

The ordering $D^i: D^{(1)} \ge D^{(2)} \ge \cdots \ge D^{(9)}$With fuzzy rule-based order statistics (,) implies the ordering $I_i: I_{(1)} \ge I_{(2)} \ge \cdots \ge I_{(9)}$. Hence in light of relation [fuzzyeq5], the vector $I_{(1)}$ is the output vector.

The proposed filter can be analyzed for computational complexity and performance analysis by using standard measures like Mean Absolute Error (MAE), Peak Signal to Noise Ratio (PSNR) and Normalized Color Difference (NCD).

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