Volume 6, No.2, FEBRUARY 2019
Journal of Global Research in Mathematical Archives
RESEARCH PAPER
Available online at http://www.jgrma.info
ISSN 2320 – 5822

SUPER FAIR DOMINATING SET IN GRAPHS

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Abstract: In this paper, we initiate the study of super fair dominating set of a graph G by giving the super fair domination number of some special graphs. Further, we shows that given positive integers k, m and n such that $n \geq 2$ and $1 \leq k \leq m \leq n-1$, there exists a connected graph G with $|V(G)| = n, \gamma_{fd}(G) = k$, and $\gamma_{sfd}(G) = m$. Finally, we characterize the super fair dominating set of the join of two graphs.

Mathematics Subject Classification: 05C69

Keywords: dominating set, fair dominating set, super dominating set, super fair dominating set

1. INTRODUCTION

Let G be a simple graph. A subset S of a vertex set $V(G)$ is a dominating set of G if for every vertex $v \in V(G) \setminus S$, there exists a vertex $x \in S$ such that xv is an edge of G. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set S of G. Dominating sets have several applications in a variety of fields, including communication and electrical networks, protection and location strategies, data structures, social networks and others. For more background on dominating sets, the reader may refer to [1, 2, 3, 4, 5, 6]. Some variants of domination in graphs are found in [7, 8, 9, 10, 11]. Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [12].

A fair dominating set in a graph G (or FD-set) is a dominating set S such that all vertices not in S are dominated by the same number of vertices from S; that is, every two vertices not in S have the same number of neighbors in S. Thus a dominating set $S \subseteq V(G)$ is an FD-set in G if for every two distinct vertices u and v from $V(G) \setminus S$, $|N(u) \cap S| = |N(v) \cap S|$. The fair domination number, $\gamma_{fd}(G)$, of G is the minimum cardinality of a FD-set. For an integer $k \geq 1$, a k-fair dominating set, abbreviated kFD-set, is a dominating set $S \subseteq V(G)$ such that $|N(u) \cap S| = k$ for every $u \in V(G) \setminus S$. The k-fair domination number of G, denoted $\gamma_{kfd}(G)$, is the minimum cardinality of a kFD-set. The concepts of fair domination and k-fair domination in graphs were introduced by Caro, Hansberg, and Henning [13].

The super dominating sets in graphs was initiated by Lemanska et.al. [14]. Variation of super domination in graphs can be read in the paper [15, 16]. A set $D \subset V(G)$ is called a super dominating set if for every vertex $u \in V(G) \setminus D$, there exists $v \in D$ such that $N_G(v) \cap (V(G) \setminus D) = \{u\}$. The super domination number of G is the minimum cardinality among all super dominating set in G denoted by $\gamma_{sp}(G)$.

Motivated by super domination and fair domination, we initiate the study of super fair domination in graphs. A fair dominating set $S \subseteq V(G)$ is a super fair dominating set(or SFD-set) if for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $N_G(v) \cap (V(G) \setminus S) = \{u\}$. The minimum cardinality of an SFD-set, denoted by $\gamma_{sfd}(G)$, is called the super fair domination number of G. For general concepts we refer the reader to [17].

2. RESULTS

Remark 2.1 A super fair dominating set is a super dominating and a fair dominating set of a nontrivial graph G.

Since the minimum super dominating set S of a nontrivial complete graph K_n is $n-1$, it follows that $\gamma_{sfd}(K_n) = n-1$. With this observation, the following remark holds.

Remark 2.2 Let G be a nontrivial connected graph G of order n. then $1 \leq \gamma_{fd}(G) \leq \gamma_{sfd}(G) \leq n-1$.

The path P_n of order n is the graph with distinct vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. In this case, P_n is also called a v_1-v_n path or the path $P(v_1, v_n)$.

Observation 2.3 *Let $n \geq 2$.*

$$\gamma_{sfd}(P_n) = \begin{cases} \frac{n}{2}, & if\ n\ is\ even \\ \frac{n+1}{2}, & if\ n\ is\ odd. \end{cases}$$

Remark 2.4 Let $n \geq 2$. Then $\gamma_{fd}(P_n) \leq \gamma_{sfd}(P_n)$ with equality occurs when $n = 2$ or $n = 4$.

The cycle C_n of order n, $n \geq 3$, is the graph with distinct vertices $v_1, v_2, \ldots, v_n$ and edges v_1v_2, $v_2v_3, \ldots, v_{n-1}v_n$, v_nv_1.

Observation 2.5 *Let $n \geq 3$.*

$$\gamma_{sfd}(C_n) = \begin{cases} 2, & if\ n = 3, \\ \frac{n}{2}, & if\ n \equiv 0(mod4), \\ \frac{n+1}{2}, & if\ n \equiv 1(mod4), \\ \frac{n+2}{2}, & if\ n \equiv 2(mod4), \\ \frac{n+3}{2}, & if\ n \equiv 3(mod4). \end{cases}$$

Remark 2.6 Let $n \geq 3$. Then $\gamma_{fd}(C_n) \leq \gamma_{sfd}(C_n)$ with equality occur when $n = 5$.

A complete graph of order n, denoted by K_n, is the graph in which every pair of its distinct vertices are joined by an edge.

Observation 2.7 Let $n \geq 2$. The $\gamma_{sfd}(K_n) = n - 1$.

Remark 2.8 Let $n \geq 2$. Then $\gamma_{fd}(K_n) \leq \gamma_{sfd}(K_n)$ with equality occur when $n = 2$.

A graph G is called a bipartite graph if its vertex-set $V(G)$ can be partitioned into two nonempty subsets V_1 and V_2 such that every edge of G has one end in V_1 and one end in V_2. The sets V_1 and V_2 are called the partite sets of G. If each vertex in V_1 is adjacent to every vertex in V_2, then G is called a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$.

Observation 2.9 *Let $m \geq 2$ and $n \geq 2$. Then*

$$\gamma_{sfd}(K_{m,n}) = \begin{cases} m + n - 1, & if\ m \neq n, \\ m + n - 2, & if\ m = n. \end{cases}$$

Remark 2.10 Let $m \geq 2$ and $n \geq 2$. Then $\gamma_{fd}(K_{m,n}) \leq \gamma_{sfd}(K_{n,m})$ with equality occur when $n = m = 2$.

A star graph $S_n = K_1 + P_n$ is a complete bipartite $K_{1,n}$ where $n \geq 1$.

Observation 2.11 $\gamma_{sfk}(S_n) = n$ for all $n \geq 1$.

Remark 2.12 Let $n \geq 1$. Then $\gamma_{fd}(S_n) \leq \gamma_{sfd}(S_n)$ with equality occur when $n = 1$.

Let $n \geq 1$. The fan of order $n + 1$, denoted by F_n, is the graph $K_1 + P_n$.

Observation 2.13 *Let $n \geq 1$. Then*

$$\gamma_{sfd}(F_n) = \begin{cases} n, & if\ n = 1\ or\ n = 3, \\ \frac{n+3}{2}, & if\ n\ is\ odd\ and\ n \geq 5. \\ \frac{n+2}{2}, & if\ n\ is\ even. \end{cases}$$

Remark 2.14} Let $n \geq 1$. Then $\gamma_{fd}(F_n) \leq \gamma_{sfd}(F_n)$ with equality occur when $n = 1$.

Let $n \geq 3$. The wheel of order $n+1$, denoted by W_n, is the graph $K_1 + C_n$.

Observation 2.15 *Let $n \geq 3$. Then*

$$\gamma_{sfd}(W_n) = \begin{cases} 3, & if \ n = 3, \\ \frac{n+2}{2}, & if \ n \equiv 0(mod4), \\ \frac{n+3}{2}, & if \ n \equiv 1(mod4), \\ \frac{n+4}{2}, & if \ n \equiv 2(mod4), \\ \frac{n+5}{2}, & if \ n \equiv 3(mod4) \ and \ n \neq 3. \end{cases}$$

Remark 2.16 $\gamma_{fd}(W_n) < \gamma_{sfd}(W_n)$ for all $n \geq 3$.

Consider the graph $G = K_1 + C_7$ with vertex set $V(G) = \{x\} \cup \{v_1, v_2, v_3, \ldots, v_7\}$ and edge set $E(G) = \{v_i v_{i+1}: i = 1,2,\ldots,6\} \cup \{v_7 v_1\} \cup \{x v_i: i = 1,2,\ldots,7\}$. The set $S = \{x, v_1, v_2, v_4, v_5\}$ is a minimum super dominating set of G but not a fair dominating set of G since $v_3, v_6 \in V(G) \setminus S$ and $|N(v_3) \cap S| \neq |N(v_6) \cap S|$. Thus the following remark holds.

Remark 2.17 Every minimum super dominating set need not be a fair dominating set in a graph G.

The following result says that the value of the parameter $\gamma_{sfd}(G)$ ranges over all positive integers $1,2,\ldots,n-1$.

Theorem 2.18 Given positive integers k, m and n such that $n \geq 2$ and $1 \leq k \leq m \leq n-1$ there exists a connected graph G with $|V(G)| = n$, $\gamma_{fd}(G) = k$ and $\gamma_{sfd}(G) = m$.

Proof: Consider the following cases:

Case1. Suppose $m = n-1$.

Let $G = K_n$. Then, clearly, $|V(G)| = n$ and $\gamma_{fd}(G) = 1 = k$ and $\gamma_{dfd}(G) = n-1 = m$.

Case2. Suppose $m < n-1$.

Consider $1 \leq k = m$. Let $G = P_k \circ K_1$. Then the set $S = V(P_k)$ is a fair dominating set and a super dominating set of G. Since S is both minimum fair and super dominating sets, it follows that S is a minimum super fair dominating set of G. Thus, $|V(G)| = 2k = n$, $\gamma_{fd}(G) = |S| = k$, and $\gamma_{sfd}(G) = k = m$.

Consider $1 < k < m$. Let $G = C_n$ where $n = 2m-1$ $(m \geq 5)$, $k \in \{n/3, (n+2)/3, (n+4)/3\}$ and $n \equiv 1(mod\ 4)$. If $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_nv_1\}$, then the set $A = \{v_{3i-2}: i = 1,2,\ldots,n/3\}$ or $A = \{v_{3i-2}: i = 1,2,\ldots,(n+2)/3\}$ or $A = \{v_{3i-2}: i = 1,2,\ldots,(n+4)/3\}$ is a minimum fair dominating set of G and the set $B = \left\{v_{4i-3}: i = 1,2,\ldots,\frac{n+3}{4}\right\} \cup \{v_{4i-2}: i = 1,2,\ldots,(n-1)/4\}$ is a minimum super fair dominating set of G. Thus, $|V(G)| = n$, $\gamma_{fd}(G) = |A| = k$, and $\gamma_{sfd}(G) = |B| = (n+3)/4 + (n-1)/4 = (n+1)/2 = (2m-1+1)/2 = m$.

Consider $1 = k < m$. Let $G = \{x\} + P_{n-1}$ where $n = 2m-1$ and $n \equiv 1(mod\ 4)$. If $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$ and $E(P_{n-1}) = \{v_1v_2, v_2v_3, \ldots, v_{n-2}v_{n-1}\}$, the set $A = \{x\}$ is the minimum fair dominating set of G and the set $B = \{x\} \cup \left\{v_{4i-2}: i = 1,2,\ldots,\frac{n-1}{4}\right\} \cup \{v_{4i-1}: i = 1,2,\ldots,(n-1)/4\}$ is a minimum super fair dominating set of G. Thus, $|V(G)| = 1 + (n-1) = n$, $\gamma_{fd}(G) = |A| = 1 = k$, and $\gamma_{sfd}(G) = |B| = 1 + (n-1)/4 + (n-1)/4 = (n+1)/2 = m$.

This proves the assertion. ■

The next result is an immediate consequence of Theorem 2.18

Corollary 2.19 The difference $\gamma_{sfd}(G) - \gamma_{fd}(G)$ can be made arbitrarily large.

The join of two graphs G and H is the graph $G + H$ with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup = \{uv: u \in V(G), v \in V(H)\}$.

We need the following results for the characterization of the super fair domination of the join of two graphs.

Lemma 2.20 Let G and H be connected non-complete graphs with $|V(G)| = |V(H)|$. If $S_G = V(G) \setminus \{a\}$ for some $a \in V(G)$, $S_H = V(H) \setminus \{b\}$ for some $b \in V(H)$, and $|N_G(a)| = |N_H(b)|$, then $S = S_G \cup S_H$ is a super fair dominating set of $G + H$.

Proof: Suppose that $S_G = V(G) \setminus \{a\}$ for some $a \in V(G)$ and $S_H = V(H) \setminus \{b\}$ for some $b \in V(H)$. Then

$$\begin{aligned} S = S_G \cup S_H &= (V(G) \setminus \{a\}) \cup (V(H) \setminus \{b\}) \\ &= \big(V(G) \cup V(H)\big) \setminus \{a, b\} \\ &= V(G + H) \setminus \{a, b\} \end{aligned}$$

Thus, $V(G + H) \setminus S = \{a, b\}$. Since G is non-complete, choose $a \in V(G)$ such that $ac \neq E(G)$ for some $c \in V(G) \setminus \{a\} = S_G$. Similarly, since H is non-complete, choose $b \in V(H)$ such that $bd \neq E(H)$ for some $d \in V(H) \setminus \{b\} = S_H$. Consider $a \in V(G + H) \setminus S$. Then there exists $d \in S$ such that $N_{G+H}(d) \cap (V(G + H) \setminus S) = \{a\}$. Consider $b \in V(G + H) \setminus S$. Then there exists $c \in S$ such that $N_{G+H}(c) \cap (V(G + H) \setminus S) = \{b\}$. Thus, S is a super dominating set of $G + H$. Now,

$$\begin{aligned} |N_{G+H}(a)| &= |N_G(a) \cup V(H)| \\ &= |N_G(a)| + |V(H)| \\ &= |N_H(b)| + |V(G)| \\ &= |N_H(b) \cup V(G)| \\ &= |N_{G+H}(b)| \end{aligned}$$

Thus, for $a, b \in V(G + H) \setminus S$, $|(N_{\{G+H\}}(a) \cap S| = |N_{G+H}(b) \cap S|$ and so, S is a fair dominating set of $G + H$. Accordingly, S is a super fair dominating set of $G + H$. ■

Theorem 2.21 Let G and H be connected non-complete graphs. Then $S = S_G \cup S_H$ is a super fair dominating set of $G + H$ where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ if and only if one of the following statements is satisfied.

(i) S_G is a super fair dominating set of G and $S_H = V(H)$.
(ii) S_H is a super fair dominating set of H and $S_G = V(G)$.
(iii) $S_G = V(G) \setminus \{w\}$ for some $w \in V(G)$, $S_H = V(H) \setminus \{z\}$ for some $z \in V(H)$, and one of the following conditions hold.

$a)$ $|V(G)| = |V(H)|$ and $|N_G(w)| = |N_H(z)|$.
$b)$ $|S_H| - |S_G| = |N_H(z)| - |N_G(w)|$.

Proof: Suppose that $S = S_G \cup S_H \subseteq V(G + H)$ is a super fair dominating set of $G + H$. Consider the following cases:

Case1: Suppose S_G is a super fair dominating set of G.

If $S_H = V(H)$, then we are done with statement (i). Suppose $S_H \neq V(H)$. Let $x \in V(H) \setminus S_H$. Then $x \in V(G + H) \setminus S$ and $xy \in E(G + H)$ for all $y \in V(G)$. Let $u \in V(G) \setminus S_G$. Now, if we assume that there exists $u' \in V(G) \setminus S_G$ distinct from u, then $u', u \in N_{G+H}(z)$ for all $z \in S_H$. Thus, $N_{G+H}(z) \cap (V(G + H) \setminus S) = \{u, u' \colon u' \in V(G) \setminus S_G\} \cup \{x \in V(H) \setminus S_H \colon x \in N_H(z)\}$ contrary to our assumption that S is a super fair dominating set of $G + H$. This means that there is only one element of $V(G) \setminus S_G$ and so $S_G = V(G) \setminus \{w\}$ for some $w \in V(G)$.

Similarly, if we assume that there exists $x' \in V(H) \setminus S_H$ distinct from x, then $x', x \in N_{G+H}(v)$ for all $v \in S_G$. Thus, $N_{G+H}(v) \cap (V(G + H) \setminus S) = \{x, x' : x' \in V(H) \setminus S_H\} \cup \{u \in V(G) \setminus S_G \colon u \in N_G(v)\}$ contrary to our assumption that S is a super fair dominating set of $G + H$. This means that there is only one element of $V(H) \setminus S_H$ and so $S_H = V(H) \setminus \{z\}$ for some $z \in V(H)$.

Now, consider $|V(G)| = |V(H)|$. If there exist $w \in V(G) \setminus S_G$ and $z \in V(H) \setminus S_H$ such that $|N_G(w)| \neq |N_H(z)|$, then

$$\begin{aligned} |N_{G+H}(w)| &= |N_G(w) \cup V(H)| \\ &= |N_G(w)| + |V(H)| \\ &\neq |N_H(z)| + |V(G)| \\ &= |N_H(z) \cup V(G)| \\ &= |N_{G+H}(z)| \end{aligned}$$

Thus, for $w, z \in V(G+H) \setminus S$, $|(N_{G+H}(w) \cap S| \neq |N_{G+H}(z) \cap S|$ contrary to our assumption that and S is a fair dominating set of $G+H$. Consequently, $|N_G(w)| = |N_H(z)|$.

Consider $|V(G)| \neq |V(H)|$. Then $|S_G| = |V(G) \setminus \{w\}| \neq |V(H) \setminus \{z\}| = S_H$. If $|N_G(w)| = |N_H(z)|$, then by following similar computations above, S is not a fair dominating set of $G+H$. Consequently, $|N_G(w)| \neq |N_H(z)|$. Since S is a fair dominating set of $G+H$,

$$\begin{aligned} |N_{G+H}(w) \cap S| &= |N_{G+H}(z) \cap S| \\ |\big(N_G(w) \cup V(H)\big) \cap S| &= |\big(N_H(z) \cup V(G)\big) \cap S| \\ |(N_G(w) \cap S) \cup (V(H) \cap S)| &= |(N_H(z) \cap S) \cup (V(G) \cap S)| \\ |N_G(w) \cup S_H| &= |N_H(z) \cup S_G| \\ |N_G(w)| + |S_H| &= |N_H(z)| + |S_G| \; where \; |S_G| \neq |S_H| \\ & \qquad and \;\; |N_G(w)| \neq |N_H(z)| \\ |S_H| - |S_G| &= |N_H(z)| - |N_G(w)|. \end{aligned}$$

This proves statement (iii).

Case2: Suppose S_H is a super fair dominating set of H.

If $S_G = V(G)$, then we are done with statement (ii).

Suppose $S_G \neq V(G)$. Then by similar argument above, statement (iii) holds.

For the converse, suppose that statement (i) is satisfied. Let $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$. Let $u \in V(G) \setminus S_G$. Since S_G is a super dominating set of G, there exists $v \in S_G$ such that $N_G(v) \cap (V(G) \setminus S_G) = \{u\}$. Since $S_H = V(H)$, $u \in V(G) \setminus S_G = \big(V(G) \cup V(H)\big) \setminus \big(S_G \cup V(H)\big) = V(G+H) \setminus S$. Thus, for all $u \in V(G+H) \setminus S$, there exists $v \in S$ such that $N_{G+H}(v) \cap (V(G+H) \setminus S) = \{u\}$, that is, S is a super dominating set of $G+H$.

Let $u, u' \in V(G) \setminus S_G$. Since S_G is a fair dominating set of G, $|N_G(u) \cap S_G| = |N_G(u') \cap S_G|$. Since $S_H = V(H)$, $u, u' \in V(G) \setminus S_G = V(G+H) \setminus S$ and $N_G(u) \cup V(H) = N_{G+H}(u)$. Thus, for all $u, u' \in V(G+H) \setminus S$,

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |\big(N_G(u) \cup V(H)\big) \cap S| \\ &= |(N_G(u) \cap S) \cup (V(H) \cap S)| \\ &= |N_G(u) \cap S| + |V(H) \cap S| \\ &= |N_G(u') \cap S| + |V(H) \cap S| \\ &= |(N_G(u') \cap S) \cup (V(H) \cap S)| \\ &= |\big(N_G(u') \cup V(H)\big) \cap S| \\ &= |N_{G+H}(u') \cap S| \end{aligned}$$

This means that S is a super fair dominating set of $G+H$.

Similarly, if statement (ii) is satisfied, then S is a super fair dominating set of $G+H$.

Now, suppose that statement (iii) $a)$ is satisfied. Then by Lemma 2.20 , S is a super fair dominating set of $G+H$. Next, suppose that statement (iii) $b)$ is satisfied. Then $|S_H| - |S_G| = |N_H(z)| - |N_G(w)|$ for some $z \in V(H)$ and for some $w \in V(G)$. Thus for all $w, z \in V(G+H) \setminus S$,

$$\begin{aligned} |S_H| - |S_G| &= |N_H(z)| - |N_G(w)| \\ |N_G(w)| + |S_H| &= |N_H(z)| + |S_G| \\ |N_G(w) \cup S_H| &= |N_H(z) \cup S_G| \\ |(N_G(w) \cap S) \cup (V(H) \cap S)| &= |(N_H(z) \cap S) \cup (V(G) \cap S)| \\ |\big(N_G(w) \cup V(H)\big) \cap S| &= |\big(N_H(z) \cup V(G)\big) \cap S| \\ |N_{G+H}(w) \cap S| &= |N_{G+H}(z) \cap S|, \end{aligned}$$

since $w \in V(G) and\ z \in V(H)$.

This shows that S is a fair dominating set in $G + H$.

Finally, let $w \in V(G) \setminus S_G$ and $z \in V(H) \setminus S_H$. Note that $V(G + H) \setminus S = \{w, z\}$. Since G is non-complete, there exists $v \in S_G$ such that $wv \neq E(G)$. Thus, there exists $v \in S$ such that $N_{G+H}(v) \cap (V(G + H) \setminus S) = \{z\}$. Since H is non-complete, there exists $v' \in S_H$ such that $zv' \neq E(H)$. Thus, there exists $v' \in S$ such that $N_{G+H}(v') \cap (V(G + H) \setminus S) = \{w\}$. This shows that S is a super dominating set of $G + H$. Accordingly, S is a super fair dominating set of $G + H$. ■

As a consequence of Theorem 2.21, we obtain the following result.

Corollary 2.22 Let G and H be connected non-complete graphs of order m and n respectively. Then $\gamma_{sfd}(G + H) = min\{\gamma_{sfd}(G) + n,\ \gamma_{sfd}(H) + m,\ m + n - 2\}$.

Proof: Let G and H be connected non-complete graphs of order m and n respectively. Suppose $S = S_G + S_H$ is a super fair dominating set of $G + H$, where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$. Then by Theorem 2.21, $\gamma_{sfd}(G + H) \leq |S| = |S_G \cup V(H)|$ where S_G is a super fair dominating set of G or $\gamma_{sfd}(G + H) \leq |S| = |S_H \cup V(G)|$ where S_H is a super fair dominating set of H or $\gamma_{sfd}(G + H) \leq |S| = |S_G \cup S_H|$ where $S_G = V(G) \setminus \{w\}$ and $S_H = V(H) \setminus \{z\}$ for some $w \in V(G)$ and $z \in V(H)$. Thus,

$$\begin{aligned}\gamma_{sfd}(G + H) &\leq |S_G \cup V(H)| \text{ for all } S_G \subset V(G)\\ &= |S_G| + |V(H)| \text{ for all } S_G \subset V(G)\end{aligned}$$

This implies that $\gamma_{sfd}(G + H) \leq \gamma_{sfd}(G) + n$, or

$$\begin{aligned}\gamma_{sfd}(G + H) &\leq |S_H \cup V(G)| \text{ for all } S_H \subset V(H)\\ &= |S_H| + |V(G)| \text{for all } S_H \subset V(H)\end{aligned}$$

This implies that $\gamma_{sfd}(G + H) \leq \gamma_{sfd}(H) + m$, or

$$\begin{aligned}\gamma_{sfd}(G + H) &\leq |(V(G) \setminus \{w\}) \cup (V(H) \setminus \{z\}|\\ &= (|(V(G)| - |\{w\}|) + (|V(H)| - |\{z\}|)\\ &= (m - 1) + (n - 1)\end{aligned}$$

Therefore, $\gamma_{sfd}(G + H) = min\{\gamma_{sfd}(G) + n,\ \gamma_{sfd}(H) + m,\ m + n - 2)\}$. ■

3.REFERENCES

[1] E. J. Cockayne, and S. T. Hedetniemi, *Towards a theory of domination in graphs*, Networks, (1977) 247-261.

[2] L. L. Kelleher and M.B. Cozzens, Dominating sets in social network graphs, Math. Social Sci., Vol. 16, no. 3 1988, 267-279.

[3] E. L. Enriquez, and S.R. Canoy, Jr., *Secure Convex Domination in a Graph*. International Journal of Mathematical Analysis, Vol. 9 (2015), no. 7, 317-325.

[4] E. L. Enriquez, and S. R. Canoy, Jr. , *On a Variant of Convex Domination in a Graph*. International Journal of Mathematical Analysis, Vol. 9, 2015, no. 32, 1585-1592.

[5] E. L. Enriquez, and S. R. Canoy, Jr. , *Restrained Convex Dominating Sets in the Corona and the Products of Graphs.* Applied Mathematical Sciences, Vol. 9, 2015, no. 78, 3867 - 3873.

[6] E.M. Kiunisala, and E. L. Enriquez, *Clique Secure Domination in Graphs.* Global Journal of Pure and Applied Mathematics. Vol. 12, No. 3 (2016), pp. 2075–2084.

[7] C. M. Loquias, and E. L. Enriquez, On Secure and Restrained Convex Domination in Graphs, International Journal of Applied Engineering Research, Vol. 11, no. 7 (2016), 4707-4010.

[8] R. T. Aunzo Jr. , and E. L. Enriquez, *Convex Doubly Connected Domination in Graphs*. Applied Mathematical Sciences, Vol. 9, (2015), no. 135, 6723-6734.

[9] E. M. Kiunisala and E. L. Enriquez, *Inverse Secure Restrained Domination in the Join and Corona of Graphs*, International Journal of Applied Engineering Research 9(2016), pp.6676-6679

[10] D. P. Salve and E. L. Enriquez, *Inverse Perfect Domination in Graphs*, Global Journal of Pure and Applied Mathematics. Vol. 12, No. 1 (2016), pp. 1-10.

[11] T. J. Punzalan and E. L. Enriquez, *Inverse Restrained domination in graphs*, Global Journal of Pure and Applied

Mathematics 3(2016) 2001-2009.

[12] O. Ore. *Theory of Graphs*. American Mathematical Society, Provedence, R.I., 1962.

[13] Y. Caro, A. Hansberg, and M.A. Henning, *Fair Domination in Graphs*. University of Haifa, 1-7, 2011.

[14] M. Lemanska, V. Swaminathan, Y.B. Venkatakrishnan, R. Zuazua, *Super Dominating sets in graphs*. Proceedings of the National Academy of Sciences, Vol. 85, 2015, no. 3, 353-357.

[15] M. P. Baldado, Jr. and E. L. Enriquez, *Super Secure Domination in Graphs*, International Journal of Mathematical Archive-8(12), 2017, pp. 145-149

[16] E. L. Enriquez, Super Restrained Domination in the Corona of Graphs, International Journal of Latest Engineering Research and Applications, Volume – 03, Issue – 05, May 2018, PP – 01-06

[17] G. Chartrand and P. Zhang, *A First Course in Graph Theory*. Dover Publication, Inc., New York, 2012.