Volume 6, No.3, March 2019
Journal of Global Research in Mathematical Archives
RESEARCH PAPER
Available online at http://www.jgrma.info
ISSN 2320 – 5822

OUTER-CONVEX DOMINATION IN THE COMPOSITION AND CARTESIAN PRODUCT OF GRAPHS

Jonecis A. Dayap

Department of Mathematics and Sciences
College of Arts and Sciences
University of San Jose-Recoletos, 6000 Cebu City, Philippines
email: dayapjonecis@yahoo.com

Enrico L. Enriquez

Department of Mathematics
School of Arts and Sciences
University of San Carlos, 6000 Cebu City, Philippines
email: enricolenriquez@yahoo.com

Abstract

Let G be a connected simple graph. A set S of vertices of a graph G is an outer-convex dominating set if every vertex not in S is adjacent to some vertex in S and $V(G) \setminus S$ is a convex set. In this paper we characterize the outer-convex dominating sets in the composition and Cartesian product of two connected graphs. It is shown that the outer-convex domination number of a composition $G[H]$ of two connected graphs $G = P_m$ $(m \geq 3)$ and $H = K_n$ $(n \geq 2)$ is equal to 2 if $m = 3$ and $n(m-4)+2$ if $m > 3$. The outer-convex domination number of the Cartesian product $G \square H$ of two non-complete connected graphs G and H depends on the outer-convex domination number of G and H.

Mathematics Subject Classification: 05C69

Keywords: outer-convex dominating set, outer-convex domination, composition, Cartesian product

1 Introduction

The theory of domination is an area in graph theory with numerous research activities. More than 1,222 papers and 75 variations of domination parameters were presented and published on this area [1].Studies on this field have been growing rapidly due to its varieties of domination parameters and wide applications. This made the essential part to the researchers' motivation in conducting a research to this particular field. One of the domination parameter is outer-convex domination which was introduced by Dayap and Enriquez in 2017 [2]. In [2], the authors characterized the outer-convex domination in the join of two graphs and outer-convex domination numbers 1 and 2. In this paper, the researchers characterize the outer-convex domination resulting from some binary operations: composition and Cartesian product. As consequences, we determine the outer-convex domination number of these graphs.

Let G be a simple connected graph. A subset S of a vertex set $V(G)$ is a *dominating set* of G if for every vertex $v \in V(G)\setminus S$, there exists a vertex $x \in S$ such that xv is an edge of G. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set S of G. Dominating sets have several applications in a variety of fields, including communication and electrical networks, protection and location strategies, data structures and others. For more background on dominating sets, the reader may refer to [3]. Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [4].

A graph G is *connected* if there is at least one path that connects every two vertices $x, y \in V(G)$, otherwise, G is *disconnected*. For any two vertices u and v in a connected graph, the distance $d_G(u, v)$ between u and v is the length of a shortest path in G. A u-v path of length $d_G(u, v)$ is also referred to as *u-v geodesic*. The *closed interval* $I_G[u, v]$ consists of all those vertices lying on a u-v geodesic in G. For a subset S of vertices of G, the union of all sets $I_G[u, v]$ for $u, v \in S$ is denoted by $I_G[S]$. Hence $x \in I_G[S]$ if and only if x lies on some u-v geodesic, where $u, v \in S$. A set S is *convex* if $I_G[S] = S$. Certainly, if G is connected graph, then $V(G)$ is convex. Convexity in graphs was studied in [5, 6, 7].

A dominating set S, which is also convex, is called a *convex dominating set* of G. The *convex domination number* $\gamma_{con}(G)$ of G is the smallest cardinality of a convex dominating set of G. A convex dominating set of cardinality $\gamma_{con}(G)$ is called a γ_{con}-*set* of G. Convex domination in graphs was studied in [8, 9, 10]. A set S of vertices of a graph G is an outer-connected dominating set if every vertex not in S is adjacent to some vertex in S and the sub-graph induced by $V(G) \setminus S$ is connected. The outer-connected domination number $\widetilde{\gamma}_c(G)$ is the minimum cardinality of the outer-connected dominating set S of a graph G. The concept of outer-connected domination in graphs was introduced by Cyman [11]. Other variations on outer-connected domination were defined and characterized in [12, 13, 14, 15].

A set S of vertices of a graph G is an outer-convex dominating set if every vertex not in S is adjacent to some vertex in S and $V(G) \setminus S$ is convex. The *outer-convex domination number* of G, denoted by $\widetilde{\gamma}_{con}(G)$, is the minimum cardinality of an outer-convex dominating set of G. An outer-convex dominating set of cardinality $\widetilde{\gamma}_{con}(G)$ will be called an $\widetilde{\gamma}_{con}$-*set*. This concept was defined in [2] and further investigated as a new variation in [16].

For general concepts we refer the reader to [17].

2 Outer-convex Domination in the Composition of Graphs

The *composition* of two graphs G and H is the graph $G[H]$ with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(x,u)(y,v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$.

Note that a non-empty subset C of $V(G[H]) = V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S} (\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$. In the following results, we shall be using this form to denote any non-empty subset C of $V(G[H])$.

The following result is due to Canoy and Garces [7]

Theorem 2.1 *Let G be a connected graph and K_n the complete graph of order n. A subset $C = \bigcup_{x \in S} (\{x\} \times T_x)$ of $V(G[K_n])$, where $S \subseteq V(G)$ and $T_x \subseteq V(K_n)$ for every $x \in S$, is convex in $G[K_n]$ if and only if S is convex in G.*

The next result is due to Labendia and Canoy [10]

Theorem 2.2 *Let G be a connected graph and K_n the complete graph of order $n \geq 2$. A subset $C = \bigcup_{x \in S} (\{x\} \times T_x)$, is a convex dominating set in $G[K_n]$ if and only if S is a convex dominating set in G.*

The next result characterizes the outer convex dominating sets in $G[K_n]$.

Theorem 2.3 *Let G be a connected non-complete graph and K_n be a complete graph of order $n \geq 2$. A subset $C = \left(\bigcup_{x \in S \setminus S'} (\{x\} \times V(K_n)) \right) \cup \left(\bigcup_{x \in S'} (\{x\} \times T_x) \right)$, is an outer convex dominating set of $G[K_n]$, where $S \subseteq V(G)$, $T_x \subseteq V(K_n)$ for every $x \in S$, and $S' = \{x \in S : xy \in E(G),\ \text{for some } y \notin S\}$ if and only if S is a dominating set and $V(G) \setminus S$ is a convex set of G and one of the following is satisfied.*

(i) $S = S'$ and $T_x = V(K_n)$ for all $x \in S$.

(ii) $S \neq S'$ and $T_x \subset V(K_n)$.

Proof: Suppose that $C = \left(\bigcup_{x \in S \setminus S'} (\{x\} \times V(K_n)) \right) \cup \left(\bigcup_{x \in S'} (\{x\} \times T_x) \right)$, is an outer convex dominating set of $G[K_n]$, $S \subseteq V(G)$, $T_x \subseteq V(K_n)$ for every $x \in S$, and $S' = \{x \in S : xy \in E(G),\ \text{for some } y \notin S\}$. Then C

is a dominating set and $V(G[K_n]) \setminus C$ is a convex set of $G[K_n]$. Write, $V(G[K_n]) \setminus C$

$$\begin{aligned}
&= [V(G) \times V(K_n)] \setminus \left(\left(\bigcup_{x \in S \setminus S'} (\{x\} \times V(K_n)) \right) \cup \bigcup_{x \in S'} (\{x\} \times T_x) \right) \\
&= [V(G) \times V(K_n)] \setminus ([(S \setminus S') \times V(K_n)] \cup [S' \times T_x]) \\
&= [V(G) \times V(K_n)] \setminus ([(S \times V(K_n)) \setminus (S' \times V(K_n))] \cup (S' \times T_x)) \\
&= [(V(G) \setminus S) \times V(K_n)] \cup [(S' \times V(K_n) \setminus T_x)] \\
&= \left(\left(\bigcup_{x \in V(G) \setminus S} (\{x\} \times V(K_n)) \right) \cup \bigcup_{x \in S'} (\{x\} \times V(K_n) \setminus T_x) \right)
\end{aligned}$$

Since $V(G[K_n]) \setminus C$ is a convex set of $G[K_n]$,

$$\left(\left(\bigcup_{x \in V(G) \setminus S} (\{x\} \times V(K_n)) \right) \cup \bigcup_{x \in S'} (\{x\} \times V(K_n) \setminus T_x) \right) \text{ is convex.}$$

By computation, it can be verified that $\left(\bigcup_{x \in V(G) \setminus S} (\{x\} \times V(K_n)) \right)$ is also convex set of $G[K_n]$. Thus, $V(G) \setminus S$ is convex by Theorem 2.1. Now, let $(x, y) \in V(G[K_n]) \setminus C$. Since C is a dominating set of $G[K_n]$, there exists $(x', y') \in C$ such that $(x, y)(x', y') \in E(G[K_n])$. To show that S is dominating, we may consider only that $(x, y) \in (V(G) \setminus S) \times V(K_n)$. Then $x \in V(G) \setminus S$ and $x \neq x'$, that is, $xx' \in E(G)$. Thus, for all $x \in V(G) \setminus S$, there exists $x' \in S$ such that $xx' \in E(G)$, that is, S is a dominating set in G. Since $S' \subseteq S$, consider first that $S' = S$. Then

$$\begin{aligned}
C &= \left(\left(\bigcup_{x \in S \setminus S'} (\{x\} \times V(K_n)) \right) \cup \bigcup_{x \in S'} (\{x\} \times T_x) \right) \\
&= \left. \bigcup_{x \in S} (\{x\} \times T_x) \right) \\
&= S \times T_x \text{ for all } x \in S.
\end{aligned}$$

Suppose that $T_x \neq V(K_n)$, then there exists $y \in V(K_n) \setminus T_x$ for all $x \in S$ such that $(x, y) \notin C$, that is, $(x, y) \in V(G[K_n]) \setminus C$. Let $(x_1, y), (x_2, y), (x_3, y) \in V(G[K_n]) \setminus C$. Since $V(G[K_n]) \setminus C$ is convex, the path (x_1, y)-(x_3, y) is an (x_1, y)-(x_3, y) geodesic. Suppose $(x_1, y)(x_2, y), (x_2, y)(x_3, y) \in E(G[K_n])$. Then there exists $y' \in T_x$ for all $x \in S$ such that $(x_1, y)(x_2, y'), (x_2, y')(x_3, y) \in E(G[K_n])$. Since (x_2, y') lies on (x_1, y)-(x_3, y) geodesic, $(x_2, y') \in I_{G[K_n]}[V(G[K_n]) \setminus C]$. But $(x_2, y') \notin V(G[K_n]) \setminus C$. Thus $I_{G[K_n]}[V(G[K_n]) \setminus C] \neq V(G[K_n]) \setminus C$, and so, $V(G[K_n]) \setminus C$ is not convex contrary to our assumption. Therefore, $T_x = V(K_n)$,. This proves statement (i). Next, if $S' \neq S$, then $T_x \neq V(K_n)$, (otherwise $S' = S$). This proves statement (ii).

For the converse, suppose that S is a dominating set and $V(G) \setminus S$ is a convex set of G and statement (i) or statement (ii) is satisfied. Let $C = \left(\left(\bigcup_{x \in S \setminus S'} (\{x\} \times V(K_n)) \right) \cup \bigcup_{x \in S'} (\{x\} \times T_x) \right)$, where $S \subseteq V(G)$, $T_x \subseteq V(K_n)$ for every $x \in S$, and $S' = \{x \in S : xy \in E(G),$ for some $y \notin S\}$. Now, we will show C is an outer-convex dominating set of $G[K_n]$. Consider first that statement (i) holds. Then $S = S'$ and $T_x = V(K_n)$

for all $x \in S$, and so,

$$\begin{aligned} C &= \left(\bigcup_{x\in S\setminus S'} (\{x\} \times V(K_n)) \right) \cup \bigcup_{x\in S'} (\{x\} \times T_x)\Bigg) \\ &= \bigcup_{x\in S'} (\{x\} \times T_x)\Bigg) \\ &= \bigcup_{x\in S} (\{x\} \times V(K_n))\Bigg) \\ &= S \times V(K_n). \end{aligned}$$

Since S is dominating set of G, there exists $x' \in S$ such that $xx' \in E(G)$ for all $x \in V(G) \setminus S$. Thus, there exist $(x', y') \in C$ such that $(x, y)(x', y') \in E(G[K_n])$ for all $(x, y) \in V(G[K_n]) \setminus C$, that is, C is a dominating set of $G[K_n]$. Now,

$$\begin{aligned} V(G[K_n]) \setminus C &= [V(G) \times V(K_n)] \setminus [S \times V(K_n)] \\ &= (V(G) \setminus S) \times V(K_n) \\ &= \bigcup_{x\in V(G)\setminus S} (\{x\} \times V(K_n)). \end{aligned}$$

Since $V(G) \setminus S$ is a convex set in G, $\bigcup_{x\in V(G)\setminus S} (\{x\} \times V(K_n))$ is a convex set in $G[K_n]$ by Theorem 2.1. Thus, $V(G[K_n]) \setminus C$ is convex in $G[K_n]$. Accordingly, C is an outer-convex dominating set in $G[K_n]$.

Next, consider that statement (ii) holds. Then

$$C = \left(\bigcup_{x\in S\setminus S'} (\{x\} \times V(K_n)) \right) \cup \left(\bigcup_{x\in S'} (\{x\} \times T_x) \right) \text{ where } T_x \neq V(K_n).$$

Since S is dominating set of G, C is a dominating set of $G[K_n]$ by similar arguments used in the proof of (i). Now,

$$\begin{aligned} V(G[K_n]) \setminus C &= [V(G) \times V(K_n)] \setminus [\bigcup_{x\in S\setminus S'} (\{x\} \times V(K_n)) \cup \bigcup_{x\in S'} (\{x\} \times T_x)] \\ &= [V(G) \times V(K_n)] \setminus [((S \setminus S') \times V(K_n)) \cup (S' \times T_x)] \\ &= [(V(G) \setminus S) \times V(K_n)] \cup [S' \times (V(K_n) \setminus T_x)] \\ &= [\bigcup_{x\in V(G)\setminus S} (\{x\} \times V(K_n))] \cup [\bigcup_{x\in S'} (\{x\} \times T_x)]. \end{aligned}$$

Since $V(G) \setminus S$ is a convex set in G, $\bigcup_{x\in V(G)\setminus S} (\{x\} \times V(K_n))$ is a convex set in $G[K_n]$ by Theorem 2.1. By computation, it can be shown that

$$[\bigcup_{x\in V(G)\setminus S} (\{x\} \times V(K_n))] \cup [\bigcup_{S'} (\{x\} \times T_x)]$$

is also convex set in $G[K_n]$. Thus, $V(G[K_n]) \setminus C$ is convex in $G[K_n]$. Accordingly, C is an outer-convex dominating set in $G[K_n]$. $\square$

The next result is a quick consequence of Theorem 2.3.

Corollary 2.4 *Let $G = P_m$ and $H = K_n$ with $m \geq 3$ and $n \geq 2$. Then*

$$\widetilde{\gamma}_{con}(G[H]) = \begin{cases} 2, & if \;\; m = 3 \\ n(m-4)+2, & if \;\; m > 3 \end{cases}$$

Proof: Suppose that S is a dominating set and $V(G)\backslash S$ is a convex set of G. Let $C = \left(\bigcup_{x\in S\backslash S'}(\{x\}\times V(H))\right)\cup \bigcup_{x\in S'}(\{x\}\times T_x)\Bigg)$, and consider $|T_x| = 1$ for every $x \in S'$, and $S' = \{x \in S : xy \in E(G),\ \text{for some } y \notin S\}$. Then C is an outer-convex dominating set of $G[H]$ by Theorem 2.3. Thus,

$$\begin{aligned}
\widetilde{\gamma}_{con}(G[H]) &\leq |C| \\
&= \left|\left(\bigcup_{x\in S\backslash S'}(\{x\}\times V(H))\right)\cup \bigcup_{x\in S'}(\{x\}\times T_x)\Bigg)\right| \\
&= |((S\setminus S')\times V(H))\cup(S'\times T_x)|, \forall x\in S' \\
&= (|S|-|S'|)\cdot|V(H)| + (|S'|\cdot|T_x|), \forall x\in S'
\end{aligned}$$

Since $V(G)\backslash S$ is convex set and S is a dominating set of $G = P_{m\geq 3}$, it follows that $|V(G)\backslash S| = |V(G)| - |S| \leq 2$, that is, $|S| \geq |V(G)| - 2 = m - 2$. This implies that $|S| = 2$ if $m = 3$, and $|S| = m - 2$ if $m > 3$. Further, since $S' = \{x \in S : xy \in E(G),\ \text{for some } y \notin S\}$, it can be readily seen that $|S'| = 2$. Thus, if $m = 3$, then $|S| = 2 = |S'|$ and so

$$\begin{aligned}
\widetilde{\gamma}_{con}(G[H]) &\leq (|S|-|S'|)\cdot|V(H)| + (|S'|\cdot|T_x|), \forall x\in S' \\
&= (2-2)\cdot n + (2\cdot 1) \\
&= 2.
\end{aligned}$$

if $m > 3$, then $|S| = m - 2$ and so

$$\begin{aligned}
\widetilde{\gamma}_{con}(G[H]) &\leq (|S|-|S'|)\cdot|V(H)| + (|S'|\cdot|T_x|), \forall x\in S' \\
&= (m-2-2)\cdot n + (2\cdot 1) \\
&= n(m-4)+2.
\end{aligned}$$

Thus,

$$\widetilde{\gamma}_{con}(G[H]) \leq \begin{cases} 2, & \text{if } m = 3 \\ n(m-4)+2, & \text{if } m > 3, \text{ inequality (1)} \end{cases}$$

Suppose C^o is a $\widetilde{\gamma}_{con}$-*set* in $G[H]$. Then there exists $|T_x^o|$ for all $x \in S'$ such that

$$\begin{aligned}
\widetilde{\gamma}_{con}(G[H]) &= |C^o| \\
&= \left|\left(\bigcup_{x\in S^o\backslash S^{o'}}(\{x\}\times V(H))\right)\cup\left(\bigcup_{x\in S^{o'}}(\{x\}\times T_x^o)\right)\right| \\
&= |((S^o\setminus S^{o'})\times V(H))\cup(S^{o'}\times T_x^o)|, \forall x\in S^{o'} \\
&= (|S^o|-|S^{o'}|)\cdot|V(H)| + (|S^{o'}|\cdot|T_x^o|), \forall x\in S^{o'}
\end{aligned}$$

if $m = 3$, then $|S^o| = 2 = |S^{o'}|$ and so

$$\begin{aligned}
\widetilde{\gamma}_{con}(G[H]) &= (|S|-|S^{o'}|)\cdot|V(H)| + (|S^{o'}|\cdot|T_x|), \forall x\in S^{o'} \\
&\geq (2-2)\cdot n + (2\cdot 1) \\
&= 2.
\end{aligned}$$

if $m > 3$, then $|S| = m - 2$ and so

$$\begin{aligned}
\widetilde{\gamma}_{con}(G[H]) &= (|S|-|S^{o'}|)\cdot|V(H)| + (|S^{o'}|\cdot|T_x|), \forall x\in S^{o'} \\
&\geq (m-2-2)\cdot n + (2\cdot 1) \\
&= n(m-4)+2.
\end{aligned}$$

Thus,

$$\widetilde{\gamma}_{con}(G[H]) \geq \begin{cases} 2, & \text{if } m = 3 \\ n(m-4)+2, & \text{if } m > 3, \text{ inequality } (2) \end{cases}$$

Therefore, by combining inequalities (1) and (2), we obtain the desired results. □

3 Outer-convex Domination in the Cartesian Product of Graphs

The *Cartesian product* of two graphs G and H is the graph $G\Box H$ with vertex-set $V(G\Box H) = V(G) \times V(H)$ and edge-set $E(G\Box H)$ satisfying the following conditions: $(x,a)(y,b) \in E(G\Box H)$ if and only if either $xy \in E(G)$ and $a = b$ or $x = y$ and $ab \in E(H)$.

The next results are due to Labendia and Canoy [10]

Theorem 3.1 [10] *Let G and H be connected graphs. A subset C of $V(G \times H)$ is a convex dominating set in $G \times H$ if and only if $C = C_1 \times C_2$ and*

(i) C_1 is a convex dominating set in G and $C_2 = V(H)$, or

(ii) C_2 is a convex dominating set in H and $C_1 = V(G)$.

Corollary 3.2 [10] *Let G and H be connected graphs of orders m and n respectively. Then $\gamma_{con}(G\Box H) = min\{m\gamma_{con}(H), n\gamma_{con}(G)\}$.*

The following result is the characterization of an outer-convex dominating set of $G\Box H$.

Theorem 3.3 *Let G and H be connected non-complete graphs. A subset C is an outer-convex dominating set of $G\Box H$ if and only if*

(i) $C = \bigcup_{v\in S}[\{v\} \times V(H)]$, where S is an outer-convex dominating set of G; or

(ii) $C = \bigcup_{v\in V(G)}[\{v\} \times T_v]$, where T_v is an outer-convex dominating set of H for each $v \in V(G)$.

Proof: Suppose that C is an outer-convex dominating set of $G\Box H$. Then $V(G\Box H) \setminus C$ is convex and C is a dominating set of $G\Box H$. Consider the following cases.

*Case*1: If $V(G\Box H) \setminus C$ is dominating, then $V(G\Box H) \setminus C = (V(G) \setminus S) \times V(H)$ where $V(G) \setminus S$ is a convex dominating set in G and $T_v = V(H)$, for all $v \in S$, by Theorem 3.1(i). This implies that

$$\begin{aligned} C &= V(G\Box H) \setminus [(V(G) \setminus S) \times V(H)] \\ &= [V(G) \times V(H)] \setminus [(V(G) \setminus S) \times V(H)] \\ &= [V(G) \setminus (V(G) \setminus S)] \times V(H) \\ &= S \times V(H). \end{aligned}$$

Since C is a dominating set of $G\Box H$, S must be a dominating set of G and hence S an outer-convex dominating set of G (since $V(G) \setminus S$ is convex). Thus, $C = S \times V(H) = \bigcup_{v\in S}[\{v\} \times V(H)]$, where S is an outer-convex dominating set of G.

*Case*2: If $V(G\Box H) \setminus C$ is not dominating, then $V(G\Box H) \setminus C = (V(G) \setminus S) \times V(H)$ where $V(G) \setminus S$ is a convex set in G and $T_v = V(H)$, for all $v \in S$. This implies that

$$\begin{aligned} C &= V(G\Box H) \setminus [(V(G) \setminus S) \times V(H)] \\ &= [V(G) \times V(H)] \setminus [(V(G) \setminus S) \times V(H)] \\ &= [V(G) \setminus (V(G) \setminus S)] \times V(H) \\ &= S \times V(H). \end{aligned}$$

By similar arguments used in Case1, $C = S \times V(H) = \bigcup_{v \in S} [\{v\} \times V(H)]$, where S is an outer-convex dominating set of G. This proves statement (i).

Further, since $V(G\Box H) \setminus C$ is convex, consider the following cases.

*Case*1: If $V(G\Box H) \setminus C$ is dominating, then $V(G\Box H) \setminus C = V(G) \times [V(H) \setminus T_v]$ where $S = V(G)$ and $V(H) \setminus T_v$, is a convex dominating set in H for all $v \in S$, by Theorem 3.1(ii). This implies that

$$\begin{aligned} C &= [V(G) \times V(H)] \setminus (V(G) \times [V(H) \setminus T_v]) \text{ for all } v \in S \\ &= V(G) \times T_v \text{ for all } v \in S. \end{aligned}$$

Since C is a dominating set of $G\Box H$, T_v must be a dominating set of H for each $v \in S$ and hence T_v an outer-convex dominating set of H (since $V(H) \setminus T_v$ is convex). Thus, $C = V(G) \times T_v = \bigcup_{v \in V(G)} [\{v\} \times T_v]$, where T_v is an outer-convex dominating set of H for each $v \in V(G)$.

*Case*2: If $V(G\Box H) \setminus C$ is not dominating, then $V(G\Box H) \setminus C = V(G) \times [V(H) \setminus T_v]$ where $S = V(G)$ and $V(H) \setminus T_v$, is a convex set in H for all $v \in S$. This implies that

$$\begin{aligned} C &= [V(G) \times V(H)] \setminus (V(G) \times [V(H) \setminus T_v]) \text{ for all } v \in S \\ &= V(G) \times T_v \text{ for all } v \in S. \end{aligned}$$

By similar arguments used in Case1, $C = V(G) \times T_v = \bigcup_{v \in V(G)} [\{v\} \times T_v]$, where T_v is an outer-convex dominating set of H for each $v \in V(G)$. This proves statement (ii).

For the converse, suppose that statement (i) is satisfied. Then, $C = \bigcup_{v \in S} [\{v\} \times V(H)] = S \times V(H)$. Since S is a dominating set of G, it follows that C is a dominating set of $G\Box H$. Now,

$$\begin{aligned} V(G\Box H) \setminus C &= [V(G) \times V(H)] \setminus [\bigcup_{v \in S} \{v\} \times V(H)] \\ &= [V(G) \times V(H)] \setminus [S \times V(H)] \\ &= [V(G) \setminus S] \times V(H) \end{aligned}$$

Since S is an outer-convex dominating set of G, it follows that $V(G) \setminus S$ is convex. Thus, $[V(G) \setminus S] \times V(H)$ is convex, that is, $V(G\Box H) \setminus C$ is convex. Accordingly, C is an an outer-convex dominating set of $G\Box H$.

Suppose that statement (ii) is satisfied. Then, $C = \bigcup_{v \in V(G)} [\{v\} \times T_v] = V(G) \times T_v$ for all $v \in V(G)$. Since T_v is a dominating set of H for each $v \in V(G)$, it follows that C is a dominating set of $G\Box H$. Now,

$$\begin{aligned} V(G\Box H) \setminus C &= [V(G) \times V(H)] \setminus [\bigcup_{v \in V(G)} \{v\} \times T_v] \\ &= [V(G) \times V(H)] \setminus [V(G) \times T_v] \text{ for all } v \in V(G) \\ &= V(G) \times [V(H) \setminus T_v] \text{ for all } v \in V(G). \end{aligned}$$

Since T_v is an outer-convex dominating set of H for all $v \in V(G)$, it follows that $V(H) \setminus T_v$ is convex. Thus, $V(G) \times [V(H) \setminus T_v]$ for all $v \in V(G)$ is convex, that is, $V(G\Box H) \setminus C$ is convex. Accordingly, C is an an outer-convex dominating set of $G\Box H$. $\Box$

The following result is an immediate consequence of Theorem 3.3

Corollary 3.4 *Let G and H be connected non-complete graphs.*

$$\widetilde{\gamma}_{con}(G\Box H) = \min\{\widetilde{\gamma}_{con}(G) \times |V(H)|, |V(G)| \times \widetilde{\gamma}_{con}(H)\}.$$

Proof: Suppose that $C = \bigcup_{v\in S}[\{v\} \times V(H)]$, where S is an outer-convex dominating set of G. Then C is an outer-convex dominating set of $G\Box H$ by Theorem 3.3(i). Thus,

$$\begin{aligned}\widetilde{\gamma}_{con}(G\Box H) &\leq |C| \\ &= |\bigcup_{v\in S}[\{v\}\times V(H)]| \\ &= |S\times V(H)| \\ &= |S|\cdot|V(H)| \text{ for all outer-convex } S \text{ of } G \\ &= \widetilde{\gamma}_{con}(G)\cdot|V(H)|, \text{ inequality (1).}\end{aligned}$$

Further, suppose that $C = \bigcup_{v\in V(G)} [\{v\} \times T_v]$, where T_v is an outer-convex dominating set of H for each $v \in V(G)$. Then C is an outer-convex dominating set of $G\Box H$ by Theorem 3.3(ii). Thus,

$$\begin{aligned}\widetilde{\gamma}_{con}(G\Box H) &\leq |C| \\ &= |\bigcup_{v\in V(G)}[\{v\}\times T_v]| \\ &= |V(G)\times T_v| \text{ for all } v\in V(G) \\ &= |V(G)|\cdot|T_v| \text{ for all } v\in V(G) \\ &= |V(G)|\cdot\widetilde{\gamma}_{con}(H), \text{ inequality (2).}\end{aligned}$$

By combining inequalities (1) and (2), we obtain the desired results. □

References

[1] Haynes, T. W., Hedetniemi, S., & Slater, P. (1998). Fundamentals of domination in graphs. *CRC Press.*

[2] Dayap, J. A., & Enriquez, E.L.(2017). *Outer-convex Domination in Graphs.* Manuscript submitted for publication.

[3] Cockayne, E. J., & Hedetniemi, S. T. (1977). Towards a theory of domination in graphs. *Networks*, 7(3), 247-261.

[4] Ore, O. (1962). *Theory of Graphs* (Vol. 38). American Mathematical Society.

[5] Harary, F., & Nieminen, J. (1981). Convexity in graphs. *Journal of Differential Geometry*, 16(2), 185-190.

[6] Chartrand, G., & Zhang, P. (1999). *Convex sets in graphs. In Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing* (Boca Raton, FL, 1999). Congr. Numer. (Vol. 136, pp. 19-32).

[7] Canoy Jr, S.R., & and Garces, I. J. L. (2002). Convex sets under some graph operations. *Graphs and Combinatorics*, 18(4), 787-793.

[8] Enriquez, E. L., & Canoy Jr, S. R. (2015). Secure convex domination in a graph. *International Journal of Mathematical Analysis*, 9(7), 317-325.

[9] M. Lemanska, Weakly convex and convex domination numbers. *Opuscula Mathematica*, 24 (2004), 181-188.

[10] Labendia, M.A., Canoy, S.R. Convex Domination in the Composition and Cartesian Product of Graphs. *Czechoslovak Mathematical Journal*, 62(2012), 1003-1009.

[11] Cyman, J. (2007). The outer^1-connected domination number of a graph. *Australasian journal of Combinatorics*, 38, 35-46.

[12] Enriquez, E.L., Fernandez, V., Punzalan, T., & Dayap, J.A. (2017). Perfect outer-connected domination in the join and corona of graphs. *Recoletos Multidisciplinary Research Journal*, 4(2).

[13] Cyman, J. (2010). Total outer-connected domination in trees. *Discussiones Mathematicae Graph Theory*, 30(3), 377-383.

[14] Favaron, O., Karami, H., & Sheikholeslami, S. M. (2014). On the total outer-connected domination in graphs. *Journal of Combinatorial Optimization*, 27(3), 451-461.

[15] Hinampas Jr, R. G., Lomarda-Hinampas, J., & Dahunan, A. R. (2017). 1-movable Independent Outer-connected Domination in Graphs. *Global Journal of Pure and Applied Mathematics*, 13(1), 41-49.

[16] Dayap, J.A., Dionsay, J.S., & Telen, R.T. (2018). Perfect outer-convex domination in graphs. *International Journal of Latest Engineering Research and Applications*, 3(7), 25-29.

[17] Chartrand, G., & Zhang, P. (2013). *A first course in graph theory.*Courier Corporation.

