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CONVEX SECURE DOMINATIONIN THE JOIN AND CARTESIAN PRODUCT OF GRAPHS

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Abstract: Let Gbe a connected simple graph. A convex dominating set S of $G$is a convex secure dominating set, if for each element $u \in V(G)\backslash S$ thereexists an element $v \in S$such that $uv \in E(G)$ and $(S\backslash\{v\}) \cup \{u\}$ is adominating set. The convex secure domination number of G, denoted by$\gamma_{cs}(G)$, is the minimum cardinality of a convex secure dominating set of G. A convex secure dominating set of cardinality$\gamma_{cs}(G)$ will be calleda γ_{cs}-set. In this paper, we investigate the concept and give some important results on convex secure dominating sets in the join and Cartesian product of two graphs.

Mathematics Subject Classification: 05C69

***Keywords*:** dominating set, convex dominating set, convex secure dominating set

1 INTRODUCTION

Let Gbe a simple connected graph. A subset S of a vertex set $V(G)$ is a dominating set of$G$ if for every vertex $v \in V(G)\backslash S$, there exists a vertex$x \in S$ such thatxv is an edge of G. The domination number$\gamma(G)$ of $G$is the smallest cardinality of a dominating set$S$ of G. Dominating sets have several applications in a variety of fields, including communication and electrical networks, protection and location strategies, data structures and others. For more background on dominating sets, the reader may refer to [3, 18]. Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [21]. A graph G is connected if there is at least one path that connects every two vertices $x, y \in V(G)$, otherwise, G is disconnected. For any two vertices u and v in a connected graph, the distance$d_G(u, v)$ between u and $v$is the length ofa shortest path in$G$. Au-v path of length $d_G(u, v)$ is also referred to as u-v geodesic. The closed interval $I_G[u, v]$ consist of all those vertices lying on au-v geodesic in$G$. For a subset $S$of vertices of$G$, the union of all sets$I_G[u, v]$for$u, v \in S$ is denoted by$I_G[S]$. Hence$x \in I_G[S]$ if and only if$x$ lies on some u-v $g$eodesic, where$u, v \in S$. A set$S$ is convex if $I_G[S] = S$. Certainly, if$G$ is connected graph, then$V(G)$ is convex. Convexity in graphs was studiedin [12, 14, 22]. Some variants of convex domination in graphs are found in[1, 6, 8, 20].

A complete graph of order n, denoted byK_n, is the graph in which everypair of its distinct vertices are joined by an edge. A nonempty subset$S$of $V(G)$is a clique in Gif the graph $\langle S\rangle$induced by S is complete. A nonempty subset S of a vertex set $V(G)$ is a clique dominating set of Gif$S$is a dominating setand S is a clique in$G$. Clique domination in a graph is found in the paper ofDaniel and Canoy [23]. Some variant of clique domination in graphs is foundin [19].

A dominating set S which is also convex is called a convex dominating set of$G$. The convex domination number $\gamma_{con}(G)$ of$G$ is the smallest cardinality of a convex dominating set of$G$. A convex dominating set of cardinality$\gamma_{con}(G)$ is called aγ_{con}-set of$G$. Convex domination in graphs has been studied in [16, 22]. A dominating set S in$G$ is called a secure dominating set in$G$ if for every $u \in V(G) \setminus S$, there exists$v \in S \setminus N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set. The minimum cardinality of secure dominating set is called the secure domination number of G and is denoted by $\gamma_s(G)$. A secure dominating set of cardinality $\gamma_s(G)$ is called s-set of G. The concept of secure dominationin graphs was studied and introduced by E.J. Cockayne et.al [4, 5, 2]. Recently,Enriquez and Canoy, introduced a new domination parameter, the concept of secure convex domination in graphs [7]. Some variants of secure domination in graphs are found in [9, 10, 11, 17].

Motivated by the definition of convex domination and secure domination in graphs, we define a new domination in a graph. A convex dominating set$S$ of$G$is a convex secure dominating set, if for each element u in $V(G)\backslash S$there exists an element$v$ in S such that $uv \in E(G)$ and $(S\backslash\{v\}) \cup \{u\}$ is a dominating set. The convex secure domination number of G, denoted by $\gamma_{cs}(G)$, is the minimum cardinality of a convex secure dominating set of G. A convex secure dominating set of cardinality $\gamma_{cs}(G)$ will be called a γ_{cs}-*set*. For general concepts we refer the reader to [13].

2 RESULTS

Remark 2.1 A convex secure dominating set of a graph G is a convex dominating set and secure dominating set of G.

Let G be a nontrivial connected graph. Since $V(G)$ is both convex and secure dominating set, it follows that $V(G)$ is a convex secure dominating set.

From the definition of convex secure dominating set, the following result is immediate.

Remark 2.2 Let G be a nontrivial connected graph. Then

(i) $\gamma(G) \leq \gamma_s(G) \leq \gamma_{cs}(G)$; and
(ii) $1 \leq \gamma_{cs}(G) \leq n$.

It is worth mentioning that the upper bound in Remark 2.2(ii) is sharp. For example, $\gamma_{cs}(C_n) = n \; for \; all \; n \geq 6$. The lower bound is also attainable as the following result shows.

Theorem 2.3 Given positive integers k and n such that $1 \leq k \leq n$, there exists a connected nontrivial graph G with $|V(G)| = n$ and $\gamma_{cs}(G) = k$.

Proof: Consider the following cases:

Case 1. Suppose $k = 1$.

Let $G = K_n$. Then, clearly, $|V(G)| = n$ and $\gamma_{cs}(G) = 1$.

Case 2. Suppose $2 \leq k \leq n$.

Let $H = K_r \; (r \geq 2)$ and $\overline{K}_m = [a_1, a_2, \ldots, a_m] \; (m \geq 1)$. Let $n = r + m$ and $k = m + 1$. Consider the graph G obtained from H by adding the edges $va_1, va_2, \ldots, va_{m-1}, va_m$ (see Figure 1).

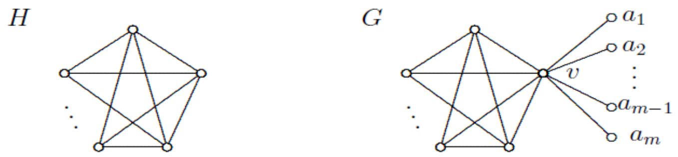

Figure 1: A graph G with $\gamma_{cs}(G) = k$

Subcase 1. Suppose that $k = 2$.

Let $m = 1$. Then the set $S = \{v, a_1\}$ is a γ_{cs}-*set* of G. Thus, $|V(G)| = r + 1 = n$ and $\gamma_{cs}(G) = 2$.

Subcase 2. Suppose that $3 \leq k \leq n - 1$.

Let $m \geq 2$. Then the set $S = \{v, a_1, a_2, \ldots, a_m\}$ is a γ_{cs}-*set* of G. Thus, $|V(G)| = r + m = n$ and $\gamma_{cs}(G) = m + 1 = k$. In particular, if $m = 2$, then $k = 3$. Further, if $r = 2$ then $k = m + 1 = m + 2 - 1 = m + r - 1 = n - 1$.

Subcase 3. Suppose $k = n$.

Let $G = C_n \, for \; all \; n \geq 6$. Then $|V(G)| = n$ and $\gamma_{cs}(G) = n$.

This proves the assertion. ■

Corollary 2.4 The difference $\gamma_{cs}(G) - \gamma(G)$ can be made arbitrarily large.

Proof: Let n be a positive integer. By Theorem 2.3, there exists a connected graph G such that $\gamma_{cs}(G) = n+1$ and $\gamma(G) = 1$. Thus, $\gamma_{cs}(G) - \gamma(G) = n$, showing that $\gamma_{cs}(G) - \gamma(G)$ can be made arbitrarily large. ■

We need the following theorems for our next results.

Theorem 2.5 [2]Let G be a graph of order $n \geq 1$. Then $\gamma_s(G) = 1$ if and only if $G = K_n$.

Theorem 2.6[9] Let G be a connected graph of order $n \geq 3$. Then $\gamma_{cs}(G) = 2$ if and only if G is non-complete and there exists distinct and adjacent vertices x and y that dominate G and satisfy one of the following conditions:

$(i) N(x) \setminus \{y\} = N(y) \setminus \{x\} = V(G) \setminus \{x, y\}$.
$(ii) \langle N(x) \setminus N[y] \rangle$ and $\langle N(y) \setminus N[x] \rangle$ are complete and for each $u \in N(x) \cap N(y)$ either $\langle (N(x) \setminus N[y]) \cup \{u\} \rangle$ or $\langle (N(y) \setminus N[x]) \cup \{u\} \rangle$ is complete.
$(iii) N(x) \setminus \{y\} = V(G) \setminus \{x, y\}$, $N(x) \setminus N[y] \neq \emptyset$ and $\langle N(x) \setminus N[y] \rangle$ is complete.

Remark 2.7 Every clique dominating set is a convex dominating set.

The converse of Remark 2.7 need not be true. For example the minimum convex dominating set of P_n for all $n \geq 5$ where $V(P_n) = \{x_1, x_2, \ldots, x_{n-1}, x_n\}$ is $S = \{x_2, x_3, \ldots, x_{n-1}\}$. But S is not a clique dominating set of P_n for all $n \geq 5$.

The following results are the characterizations of dominating sets with convex secure domination numbers of one and two.

Theorem 2.8 Let G be a graph of order $n \geq 1$. Then $\gamma_{cs}(G) = 1$ if and only if G is a complete graph.

Proof: Suppose that $\gamma_{cs}(G) = 1$. Let $S = \{v\}$ be a γ_{cs}-set in G. Then by Remark 2.1 S is a secure dominating set of G. Hence, G is a complete graph by Theorem 2.5.

For the converse, suppose that G is a complete graph. Then $\gamma_s(G) = 1$ by Theorem 2.5. Let $S = \{x\}$ be a minimum secure dominating set of G. Since S is convex set, it follows that S is a convex secure dominating set of G. Thus, $\gamma_{cs}(G) = 1$. ■

Theorem 2.9 Let G be a connected graph of order $n \geq 3$. Then $\gamma_{cs}(G) = 2$ if and only if G is non-complete and there exists distinct and adjacent vertices x and y that dominate G and satisfy one of the following conditions:

$(i) N(x) \setminus \{y\} = N(y) \setminus \{x\} = V(G) \setminus \{x, y\}$.
(ii) $\langle N(x) \setminus N[y] \rangle$ and $\langle N(y) \setminus N[x] \rangle$ are complete and for each $u \in N(x) \cap N(y)$ either $\langle (N(x) \setminus N[y]) \cup \{u\} \rangle$ or $\langle (N(y) \setminus N[x]) \cup \{u\} \rangle$ is complete.
$(iii) N(x) \setminus \{y\} = V(G) \setminus \{x, y\}, N(x) \setminus N[y] \neq \emptyset$ and $\langle N(x) \setminus N[y] \rangle$ is complete.

*Proof:*Suppose that $\gamma_{cs}(G) = 2$. Then G in non-complete by Theorem 2.8. Let $S = \{x, y\}$ be a minimum convex secure dominating set of G. Then x and y are distinct and adjacent vertices that dominate G. This implies that $y \in N(x)$ and $x \in N(y)$. Since $n \geq 3, N(x) \setminus \{y\} \neq \emptyset$ and $N(y) \setminus \{x\} \neq \emptyset$. Consider the following cases:

*Case*1. Suppose that x dominate G and y dominate G.

Let $z \in N(x) \setminus \{y\}$. Since y dominate G, $z \in N(y)$ and hence $z \in N(y) \setminus \{x\}$. Thus, $N(x) \setminus \{y\} \subseteq N(y) \setminus \{x\}$. Let $w \in N(y) \setminus \{x\}$. Similarly, since x dominate $G, N(y) \setminus \{x\} \subseteq N(x) \setminus \{y\}$. This implies that $N(x) \setminus \{y\} = N(y) \setminus \{x\}$. Further, if $z \in N(x) \setminus \{y\}$, then $z \in V(G)$ and $z \notin \{x, y\}$ implies that $z \in V(G) \setminus \{x, y\}$, that is, $N(x) \setminus \{y\} \subseteq V(G) \setminus \{x, y\}$. Now, let $u \in V(G) \setminus \{x, y\}$. Since x dominate $G, u \in N(x)$ and hence $u \in N(x) \setminus \{y\}$. Thus, $V(G) \setminus \{x, y\} \subseteq N(x) \setminus \{y\}$, that is, $N(x) \setminus \{y\} = V(G) \setminus \{x, y\}$. Therefore, $N(x) \setminus \{y\} = N(y) \setminus \{x\} = V(G) \setminus \{x, y\}$. This proves statement (i).

*Case*2. Suppose that x dominate G and y does not dominate G.

Since x dominate $G, N(x) \setminus \{y\} = V(G) \setminus \{x, y\}$ by proof in Case1. Since $n \geq 3, N(x) \setminus N[y] \neq \emptyset$. Suppose that there exists $u, v \in N(x) \setminus N[y]$ such that $uv \notin E(G)$. Then $S_u = (S \setminus \{x\}) \cup \{u\} = \{y, u\}$. Since $yv, uv \notin E(G)$, it follows that S_u is not a dominating set of G contrary to our assumption that S is a secure dominating set of G. Therefore $\langle N(x) \setminus N[y] \rangle$ must be a complete sub-graph.This proves statement (iii).

*Case*3. Suppose that neither x nor y dominate G.

By following similar arguments in Case2, $\langle N(x) \setminus N[y] \rangle$ and $\langle N(y) \setminus N[x] \rangle$ are complete graphs. Suppose that $N(x) \cap N(y) \neq \emptyset$. Let $u \in N(x) \cap N(y)$. Suppose there exists $z \in N(x) \setminus N[y]$ such that $zu \notin E(G)$. Then $S_u = (S \setminus \{y\}) \cup \{u\} = \{x, u\}$. Since S is a secure dominating set, S_u is a dominating set of G. This implies that for each $v \in N(y) \setminus \{x\}, uv \in E(G)$.

Since $\langle N(y) \setminus N[x]\rangle$ is complete, it follows that $\langle (N(y) \setminus N[x]) \cup \{u\}\rangle$ is complete. Similarly, if there exists $z \in N(y) \setminus N[x]$ such that $zu \notin E(G)$, then $\langle (N(x) \setminus N[y]) \cup \{u\}\rangle$ must be complete. This proves statement (ii).

For the converse, suppose that statement (i) or (ii) or (iii) is satisfied. Then by Theorem 2.5, $\gamma_{cls}(G) = 2$. Let $S = \{x, y\}$ be the minimum clique secure dominating set of G. Since every clique dominating set is a convex dominating set by Remark 2.7, it follows that S is a convex secure dominating set of G. Thus $\gamma_{cs}(G) \leq 2$. Since G is non-complete, $\gamma_{cs}(G) \geq 2$by Theorem 2.6. Therefore$\gamma_{cs}(G) = 2$. ■

The join of two graphs G and $H$is the graph $G + H$with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\}$.

We need the following result for the characterization of the convex secure dominating sets in the join of two graphs.

Theorem 2.10 [9] Let G and H be connected non-complete graphs. Then a proper subset S of $V(G + H)$ is a clique secure dominating set in $G + H$ if and only if one of the following statements holds:

(i) S is a clique secure dominating set of G and $|S| \geq 2$.
(ii) S is a clique secure dominating set of H and$|S| \geq 2$.
(iii) $S = S_G \cup S_H$ where$S_G = \{v\} \subset V(G)$ and$S_H = \{w\} \subset V(H)$ and
(a) $S_G$is a dominating set of G and$S_H$is a dominating set of H; or
$(b)$$S_G$ is dominating set of G and$(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H; or
$(c)$$S_H$ is dominating set of H and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G; or
$(d)$$(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G and$(V(H) \setminus S_H)\setminus N_H(S_H)$ is a clique in H.
(iv) $S = S_G \cup S_H$where S_G is a clique in G $(|S_G| \geq 2)$ and$S_H = \{w\} \subset V(H)$ and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G.
(v) $S = S_G \cup S_H$where $S_G = \{v\} \subset V(G)$ and S_H is a clique in H $(|S_H| \geq 2)$ and$(V(H)\setminus S_H) \setminus N_H(S_H)$ is a clique in H.
$(vi)$$S = S_G \cup S_H$ where$S_G$is a clique in G $(|S_G| \geq 2)$ and$S_H$is a clique in H $(|S_H| \geq 2)$.

The next result is the characterization of a convex secure dominating set in the join of two graphs.

Theorem 2.11 Let G and H be connected non-complete graphs. Then a proper subset S of $V(G + H)$ is a convex secure dominating set in $G + H$ if and only if S is a clique secure dominating set in $G + H$.

Proof: Suppose that a proper subset S of $V(G + H)$ is a convex secure dominating set in $G + H$. Consider the following cases:

Case1. Suppose that $S \cap V(H) = \emptyset$ or $S \cap V(G) = \emptyset$.

If $S \cap V(H) = \emptyset$ the $S \subseteq V(G)$. This implies that S is a convex secure dominating set of G. Now suppose that $|S| = 1$, say $S = \{a\}$. Since S is a convex secure dominating set of $G + H$, $\{z\}$ is a dominating set of $G + H$ (and hence in H) for every $z \in V(H)$. This implies that H is a complete graph, contrary to our assumption. Thus, $|S| \geq 2$. In view of Theorem 2.10(i), S is a clique secure dominating set in $G + H$. Similarly, if $S \cap V(G) = \emptyset$, then S is a clique secure dominating set in $G + H$ by Theorem 2.10(ii).

*Case*2. Suppose that $S_G = S \cap V(G) \neq \emptyset$ and $S_H = S \cap V(H) \neq \emptyset$. Then $S = S_G \cup S_H$. Consider the following subcases.

*Subcase*1. Suppose that $S_G = \{v\} \subset V(G)$ and $S_H = \{w\} \subset V(H)$. If$S_G$is a dominating set of G and$S_H$is a dominating set of H, then S is a clique secure dominating set in $G + H$ by Theorem 2.10$(iiia)$. Suppose that S_G is a dominating set of G and $S_H$is not a dominating set of H. Let $x \in (V(H) \setminus S_H) \setminus N_H(S_H)$. Since S is a convex secure dominating set of $G + H$, $\{w, x\}$ is a dominating set in $G + H$ (and hence in H). Since $wx \notin E(H)$, $xy \in E(H)$ for every $y \notin N_H(w)$. This implies that $y \in (V(H) \setminus S_H) \setminus N_H(S_H)$. Since x was arbitrarily chosen, it follows that the subgraph $\langle (V(H) \setminus S_H) \setminus N_H(S_H)\rangle$ induced by$(V(H) \setminus S_H) \setminus N_H(S_H)$ is complete. Hence,$(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H. This shows that S is a clique secure dominating set in $G + H$ by Theorem 2.10$(iiib)$. Similarly, ifS_H is dominating set of H and S_G is not a dominating set of G, then $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G. This shows that S is a clique secure dominating set in $G + H$ by Theorem 2.10$(iiic)$. If$S_G$is not a dominating set of G and $S_H$is not a dominating set of H, then by following similar arguments in $(iiib)$ and $(iiic)$, S is a clique secure dominating set in $G + H$ by Theorem 2.10$(iiid)$.

*Subcase*2. Suppose that $S_G$is a clique in G $(|S_G| \geq 2)$ and$S_H = \{w\} \subset V(H)$. If S_G is a dominating set of G, then S is a clique secure dominating set in $G + H$ by Theorem 2.10(i). Suppose that S_G is not a dominating set of G. Let $x \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since S is a convex secure dominating set of $G + H$, $S_x = (S \setminus \{w\}) \cup \{x\}$ is a dominating set of $G + H$ (and hence of G). Since $vx \notin E(G)$ for every $v \in S_G$, $xy \in E(G)$ for every $y \notin N_G(S_G)$ (otherwise, S_x is not dominating set of $G + H$. This implies that $y \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since x was arbitrarily chosen, it follows that the subgraph $\langle (V(G) \setminus S_G) \setminus N_G(S_G)\rangle$ induced by $(V(G) \setminus S_G) \setminus N_G(S_G)$ is complete. Hence $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G. This implies that S is a clique secure dominating set in $G + H$ by Theorem 2.1(iv).Similarly, if $S_G = \{v\} \subset V(G)$ and $S_H \subseteq V(H)$ $(|S_H| \geq 2)$, then S is a clique secure dominating set in $G + H$ by Theorem (v).

*Subcase*3. Suppose that S_G is a clique in G and $S_H$is a clique in H. Let$|S_G| \geq 2$. IfS_G is a dominating set of G, then S is a clique secure dominating set in $G + H$ by Theorem 2.10(i). Suppose that$S_G$is not a dominating set of G. If $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G, then S is a clique secure dominating set in $G + H$ by Theorem 2.10(iv). Suppose that $(V(G) \setminus S_G) \setminus N_G(S_G)$ is not a clique in G. If $|S\_H| = 1$, say $S_H = \{w\}$, then there exists $x \in (V(G) \setminus S_G) \setminus N_G(S_G)$ such that$S_x = (S \setminus \{w\}) \cup \{x\}$ is not a dominating set of G(and hence of $G + H$). This contradict to our assumption that S is a convex secure dominating set of $G + H$. Thus, $|S_H| \geq 2$. Similarly, if $|S_H| \geq 2$ and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is not a clique in H, then $|S_G| \geq 2$. Thus, S is a clique secure dominating set in $G + H$ by Theorem 2.10(vi).

For the converse, suppose S is a clique secure dominating set in $G + H$. Then S is a clique dominating set and a secure dominating set in $G + H$. Since every clique dominating set is a convex dominating set by Remark 2.7, it follows that S is a convex dominating set and secure dominating set of G. Accordingly, S is a convex secure dominating set of a graph $G + H$ by Remark 2.7. ■

The following result is an immediate consequence of Theorem 2.11.

Corollary 2.12 Let G and H be connected non-complete graphs.

$$\gamma_{CS}(G+H)\begin{cases}2 & if \quad \gamma_{cl}(G) = 2 \ or \ \gamma_{cl}(H) = 2, \\ 3 & if \quad S_G \ and \ V(G)\setminus N_G(S_G) \ are \ cliques \ in \ G \ or \ S_H \ and \ V(H)\setminus N_H(S_H) \ are \ cliques \ in \ H, \\ 4 & if \quad S_G \ \ is \ clique \ in \ G \ and \ S_H \ is \ clique \ in \ H,\end{cases}$$

where $(|S_G| \geq 2)$ and $(|S_H| \geq 2)$.

Remark 2.13A clique secure dominating set S of a graph G is a clique and a secure dominating set of G.

Remark 2.14 Every secure clique dominating set of a graph G is a clique secure dominating set of G.

The converse of Remark 2.14 is not true. Consider the graph in Figure 2. .

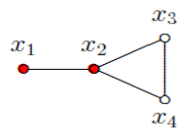

Figure 2: A graph G with $\gamma_{cls}(G) = 2$

The set $S = \{x_1, x_2\}$ is a clique secure dominating set but not a secure clique dominating set of a graph G. In fact G has no secure clique dominating set.

The Cartesian product of two graphs G and H is the graph $G \boxdot H$ with vertex-set $V(G \boxdot H) = V(G) \times V(H)$ and edge-set $E(G \boxdot H)$ satisfying the following conditions: $(x, a)(y, b) \in E(G \boxdot H)$ if and only if either $xy \in E(G)$ and $a = b$ or $x = y$ and $ab \in E(H)$.

We need the following results for the characterization of convex secure dominating sets in the Cartesian product of two graphs.

Theorem 2.15 [15] Let G and H be connected graphs. A subset C of $V(G \boxdot H)$ is a convex dominating set in $G\setminus\boxdot H$ if and only if $C = C_1 \times C_2$ and

$(i)C_1$ is a convex dominating set in G and$C_2 = V(H)$, or
$(ii)C_2$ is a convex dominating set in H and$C_1 = V(G)$.

Corollary 2.16 [15] Let G and H be connected graphs of orders m and n respectively. Then $\gamma_{con}(G \boxdot H) = min\{m \cdot$ *γcon(H),n·γcon(G)}.*

The next result is the characterization of convex secure dominating sets in the Cartesian product of two graphs.

Theorem 2.17 Let G and H be noncomplete connected graphs. A nonempty subset $C = \cup_{x\in S}(\{x\} \times T_x)$is a convex secure dominating set in $G \boxdot H$ if and only if one of the following statement is satisfied.

(i) S is a convex secure dominating set in G and $T_x = V(H)$ for each $x \in S$.
(ii) $S = V(G)$ andT_x is a convex secure dominating set of H for each $x \in S$.

Proof: Suppose that a nonempty subset $C = \cup_{x\in S}(\{x\} \times T_x)$ is a convex secure dominating set in $G \boxdot H$. Then C is a convex dominating set in $G \boxdot H$ by Remark 2.1. This implies that statement Theorem 2.15(i) or Theorem 2.15(ii) holds. Suppose that Theorem 2.15(i) holds. Let S is a convex dominating set of G and $T_x = V(H)$ for each $x \in S$. Suppose that S is not a secure dominating set in G. Then, either S or $(S \setminus \{v\}) \cup \{v\}$ is not a dominating set in G. If S is not a dominating set in G, then there exists $u \in V(G) \setminus S$ such that $uv \notin E(G)$ for all $v \in S$. Let $a \in V(H)$. Then $(u,a) \in V(G \boxdot H) \setminus C$ and $(v,a) \in C$. Since $uv \notin E(G)$ for all $v \in S$, it follows that $(u,a)(v,a) \notin E(G \boxdot H)$ for all $(v,a) \in C$. Hence, C is not a dominating set in $G \boxdot H$ contrary to our assumption. If $S_u = (S \setminus \{v\}) \cup \{u\}$ is not a dominating set in G, then there exists $u' \in V(G) \setminus S_u$ such that $u'v' \notin E(G)$ for all$v' \in S_u$. Let $C' = S_u \times V(H)$ and let$a' \in V(H)$. Then $(u',a') \in V(G \boxdot H) \setminus C'$ and $(v',a') \in C'$. Since $u'v' \notin E(G)$ for all $v' \in S_u$, it follows that $(u',a')(v',a') \notin E(G \boxdot H)$ for all$(v',a') \in C'$. This implies that C' is not a dominating set in $G \boxdot H$ and hence C is not a secure dominating set in $G \boxdot H$ contrary to our assumption that C is a convex secure dominating set in $G \boxdot H$. Thus, S must be a secure dominating set in G. This proves statement (i). Similarly, if Theorem 2.15(ii) holds, then statement (ii) holds.

For the converse, suppose that statement (i) or (ii) holds. First, suppose that statement (i) holds. Then S is a convex dominating set in G and $T_x = V(H)$ for each $x \in S$. Thus, C is a convex dominating set in $G \boxdot H$ by Theorem 2.15. Since S is a secure dominating set in G, for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and $S_u = (S \setminus \{v\}) \cup \{u\}$ is a dominating set in G. This implies that for every $u' \in V(G) \setminus S_u$, there exists $v' \in S_u$such that$u'v' \in E(G)$. Let$C' = S_u \times V(H)$ and let $a \in V(H)$. Then for every $(u',a') \in V(G \boxdot H) \setminus C'$ there exists $(v',a') \in C'$ such that$(u',a')(v',a') \in E(G \boxdot H)$. This implies that C' is a dominating set. Accordingly, C is convex secure dominating set in $G \boxdot H$. Similarly, if statement (ii) holds, then C is convex secure dominating set in $G \boxdot H$. ■

The following is a quick consequence of Theorem 2.17.

Corollary 18 Let G and H be non-complete connected graphs. Then$\gamma_{cs}(G \boxdot H]) = \{|V(G)|\gamma_{cs}(H), \gamma_{cs}(G)|V(H)|\}$.

3.CONCLUSION

An convex secure dominating set is a new variant of domination in graphs. Hence, this paper is a contribution to the development of domination theory in general. Since this is new, further investigations on binary operations and bounds of this parameter must be done to come up with substantial results. Thus, we initiate the study of the join and Cartesian product of two graphs of the convex secure dominating sets. Its corresponding convex secure domination number was also studied. From the results, we showed that the convex secure domination number of the join of two connected non-complete graphs is 2 if $\gamma_{cl}(G) = 2$ or$\gamma_{cl}(H) = 2$; 3 $S_G$and $V(G)\backslash N_G(S_G)$are cliques in G or$S_H$and $V(H)\backslash N_H(S_H)$are cliques in H; and 4 if S_G is clique in G and $S_H$is clique in H, where$(|S_G| \geq 2)$and $(|S_H| \geq 2)$. Moreover, the convex secure domination number of the Cartesian product of two connected non-complete graphs$\gamma_{cs}(G \boxdot H])$, is $\{|V(G)|\gamma_{cs}(H), \gamma_{cs}(G)|V(H)|\}$.

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