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KD PROBLEM FOR A QUASILINEAR EQUATION OF AN ELLIPTIC TYPE WITH TWO LINES OF DEGENERATION

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Abstract: The unique solvability of the KD problem for an equation of the elliptic type with two lines of degeneration is proved. The uniqueness of the solution to the KD problem is proved using the extremum principle for elliptic equations, and the existence of the method of contraction mappings.

INTRODUCTION

The research of quasilinear equations of elliptic, hyperbolic and mixed types are important not only theoretically and but also practically. Boundary problems for such equations with one line of degeneracy were studied in [1,2]. However, boundary value problems for equations with two lines of degeneracy are relatively unknown. We note the work [3,4]. The research is devoted to the study of the boundary problem for a quasilinear equation of elliptic type with two lines of degeneracy.

A FAMILY OF OPERATOR MATRICES AND MAIN RESULTS

Consider the equation

$$y^m u_{xx}+x^m u_{yy} = f(x,y,u),\ \mathrm{m} = \mathrm{const} > 0. \qquad (1)$$

Let Ω be a finite simply connected region bounded by a smooth curve σwhich ends at points A(1,0), B(0,1) and segments OA: y = 0 of the axis and OB: x = 0.

We introduce the notation

$\mathrm{P}=\{(x,y)(x,y) \in \Omega,\ -\infty < u < +\infty\}$,

$I_1 = \{(x,y) : 0 \le x \le 1,\ y = 0\}$, $I_2 = \{(x,y)\ : 0 \le y \le 1,\ x = 0\}$,

$2\mathrm{p} = \mathrm{m} + 2, 2\beta = \mathrm{m}/(\mathrm{m} + 2)$.

Further, with respect to the curve σ, we assume that:

1) let the parametric equations of the curve σ be x = x (s), y = y (s);

2) the functions x(s) and y(s) have continuous derivatives x'(s) and y'(s) on the interval [0, l], which do not vanish simultaneously, the derivatives x"(s) and y"(s) satisfy the Hölder condition of order λ_0 ($0 <\lambda_0<1$) on [0, l], where l is the length of the arc calculated from point A (1,0);

3) in a neighborhood points A(1,0) and B(0,1), the following conditions are true

$$x^{\frac{m}{2}}\left|\frac{dx}{ds}\right| \le const y^{m+1}(s),\ y^{\frac{m}{2}}\left|\frac{dy}{ds}\right| \le const x^{m+1}(s),$$

moreover$x(1) = y(0) = 1,\ x(0) = y(1) = 1$.

Definition: By the regular solution of equation (1) in the domain Ω we mean the function $u(x,y) \in C(\overline{\Omega}) \cap \mathrm{C}^1(\Omega \cup \sigma) \cap C^2(D)$, satisfying equation (1) in Ω having a bounded second derivative in ∂Ω, except for the points O(0,0) and A(1,0), B (0,1) at which they can go to infinity of order less than unity and λ_1, respectively, where λ_1 is enough a small positive number and $0 < \lambda_1 < \lambda$, где λ where λ is given throughout the paper.

Problem KD. Find a function u(x, y) from the class $C(\overline{\Omega}) \cap \mathrm{C}^1(\Omega \cup \sigma) \cap C^2(D)$, satisfying the following boundary conditions:

$$A_s[u]|_\sigma = \varphi(s), 0 < s < l, \qquad (2)$$

$$u(x,y)|_{\mathrm{OA}} = \psi(x), x \in I_1, \qquad (3)$$

$$u(x,y)|_{\mathrm{OB}} = g(y), y \in I_2, \qquad (4)$$

whereφ(s), ψ(x), g(y) are given sufficiently smooth functions, and ψ (0) = g (0),

$$A_s[u] = y^m \frac{dy}{ds}\frac{\partial u}{\partial x} - x^m \frac{dx}{ds}\frac{\partial u}{\partial y}.$$

We assume that the right-hand side of equation (1) satisfies the condition

$$f(x,y,u) = (xy)^m f_1(x,y,u)\ , \qquad (5)$$

where the function f_1 (x, y, u) is continuous and has continuous first-order derivatives with respect to all arguments in P and σ vanishes on the order 1 + λ, where λ is a sufficiently small number (which was introduced at the beginning of the article) and

$$\max_{P}\{|f_1|,|f_{1u}|\} \le const.$$

We note that boundary problems for quasilinear equations of elliptic and mixed types with one line of degeneracy were studied in [1,2].

Theorem. If condition (5) is satisfied and $f_u(x,y,u) \ge 0$ in P, then the **KD** problem for equation (1) cannot have more than one solution.

Proof. Let exist two the solution $u_1(x,y)$ и $u_2(x,y)$. Then their difference $v(x,y) = u_1(x,y) - u_2(x,y)$ will satisfy the equation:

$$y^m v_{xx} + x^m v_{yy} = f(x,y,u_1) - f(x,y,u_2) \quad (6)$$

with homogeneous boundary conditions

$$A_s[u]|_\sigma = 0,\ 0 < s < l, \quad (7)$$

$$\mathrm{u(x,y)}|_{\mathrm{OA}} = 0,\ x \in I_1, \quad (8)$$

$$\mathrm{u(x,y)}|_{\mathrm{OB}} = 0, y \in I_2. \quad (9)$$

From condition (5), the right-hand side of equation (6) can be written as

$$f(x,y,u_1) - f(x,y,u_2) = \tilde{f}_u v, \quad (10)$$

where

$$\tilde{f}_u = \int_0^1 f(x,y,tu_1 + (1-t)v)dt.$$

Then, by (10), equation (6) takes the form

$$y^m v_{xx} + x^m v_{yy} - \tilde{f}_u v = 0.$$

Since $f_u \ge 0$, it follows from the extremum principle for elliptic equations that $v(x,y)$ is a positive maximum and a negative minimum takes on σ∪OA∪OB. From this, taking into account (7) - (9), we obtain $v(x,y) \equiv 0$, which means $u_1(x,y) = u_2(x,y)$.

We turn to the proof of the existence of a solution to the problem **KD**.

The solution of the **KD** problem for equation (1) in the domain Ω with boundary conditions (2) - (4) by the known method [5] is equivalently reduced to the integro-differential equation:

$$u(x,y) = \int_0^1 t^m \psi(t) \frac{\partial}{\partial z} g_4^*(t,0;x,y)dt + \int_0^1 z^m g_4^*(0,z;x,y)dt + \int_0^l \varphi(s) g_4^*(\xi(s),\eta(s);x,y)ds +$$

$$+ \iint_\Omega g_4^*(\xi,\eta;x,y) f(\xi,\eta,u) d\eta d\xi, \quad (11)$$

where

$$g_4^*(\xi,\eta;x,y) = g_{04}(\xi,\eta;x,y) + H_4^*(\xi,\eta;x,y),$$

$$H_4^*(\xi,\eta;x,y) = \int_0^l \mu_4^*(s;\xi,\eta) A_s^+[g_{04}(\xi_1,\eta_1;x,y)]ds,$$

here $\mu_4^*(s;\xi,\eta)$–solution of integralequation

$$\mu_4^*(s;\xi,\eta) + 2\int_0^l K_4^*(s,t)\mu_4^*(t;\xi,\eta)dt = -2q_4(\xi_1(s),\eta_1(s);\xi,\eta),$$

$$K_4^*(s,t) = A_s^+\left[q_4\left(\xi_1(t),\eta_1(t);x(s),y(s)\right)\right],$$

where $g_{04}(\xi,\eta;x,y)$ is the Greena function of the Dirichlet problem for equation (1) with $f(x,y,u) = 0$ in the normal domain Ω_0^0, where the domain Ω_0^0 is bounded by the normal curve σ_0: $x^{2p} + y^{2p} = 1$ and has the form:

$$g_{04}(\xi,\eta;x,y) = q_4(\xi,\eta;x,y) - \left(\frac{1}{\bar{R}_0^2}\right)^{2\beta} q_4(\xi,\eta;\bar{x},\bar{y}) \quad (12)$$

$$\bar{R}_0^2 = \frac{1}{p^2}x^{2p} + \frac{1}{p^2}y^{2p}, \qquad \bar{x}^p = \frac{1}{\bar{R}_0^2}x^p, \quad \bar{y}^p = \frac{1}{\bar{R}_0^2}y^p$$

where $q_4(\xi,\eta;x,y)$ - fundamental solution to the equation (1) [6,7,8].

$$q_4(\xi,\eta;x,y) = \kappa_4 (r^2)^{2\beta-2} x x_0 y y_0 F_2(2\beta-2, 1-\beta, 1-\beta, 2-2\beta; 2-2\beta; \sigma_1,\sigma_2),$$

$$\kappa_4 = \frac{1}{4\pi}\left(\frac{2}{p}\right)^{2-2\beta} \frac{\Gamma^2(1-\beta)}{\Gamma(2-2\beta)},$$

$$\sigma_1 = \frac{r^2 - r_1^2}{r^2},\ \sigma_2 = \frac{r^2 - r_2^2}{r^2}, \qquad 2\beta = \frac{m}{m+2}, \qquad r^2 = \left(\frac{1}{p}x^p - \frac{1}{p}x_0^p\right)^2 + \left(\frac{1}{p}y^p - \frac{1}{p}y_0^p\right)^2,$$

$$\left(\frac{1}{p}x^p \pm \frac{1}{p}x_0^p\right)^2 + \left(\frac{1}{p}y^p \mp \frac{1}{p}y_0^p\right)^2 = \begin{cases} r_1^2, \\ r_2^2, \end{cases}$$

$F_2(2\beta-2, 1-\beta, 1-\beta, 2-2\beta; 2-2\beta; \sigma_1,\sigma_2)$ – two variable hypergeometric function.

Let σ coincide with the normal curve σ_0: $y^{2p} + x^{2p} = 1$. Without loss of generality, we can assume that $\varphi(s) = \psi(x) = g(y) = 0$. Then equation (11) has the form:

$$u(x,y) = -\iint\limits_{\Omega} g_{04}(\xi,\eta\,;x,y\,)f(\xi,\eta,u)d\eta d\xi. \qquad (13)$$

For the Greena's function (12), we have the estimates

$$|g_{04}(\xi,\eta;x,y)| \leq \mathrm{C}\frac{(1+|\ln\sigma_1\sigma_2|)}{(r_1^2r_2^2)^{\beta}} \qquad (14)$$

where

$$\left(\frac{1}{p}x^p \pm \frac{1}{p}x_0^p\right)^2 + \left(\frac{1}{p}y^p \mp \frac{1}{p}y_0^p\right)^2 = \begin{cases} r_1^2 \\ r_2^2 \end{cases},$$

C = const, dependent on known parameters.

Solutions of equation (5) will be sought by the method of successive approximations. For the zeroth approximation, we take $u_0(x,y) = 0$.

Then if the nth approximation is defined, then we find the (n + 1) - approximation by the formula

$$u_{n+1}(x,y) == -\iint\limits_{\Omega} g_{04}(\xi,\eta\,;x,y\,)f(\xi,\eta,u_n(\xi,\eta))d\eta d\xi \qquad (15)$$

where $n = 0,1,2,3 \ldots$

Lemma. If condition (5) is satisfied, then

$$|u_{n+1}-u_n| \leq \frac{20}{m+2}MC\left(\frac{60N_1C}{m+2}\right)^n, \qquad (16)$$

where, $n = 0,1,2,3 \ldots$, $M = max|\, f(x,y\,,0),|\; N_1 = max|f_{1u}\,|$.

Proof. Let n = 0. By conditions (5) and (14), from (15) we obtain

$$|u_1-u_0| \leq \left|\iint\limits_{\Omega} g_{04}(\xi,\eta;x,y\,)f(\xi,\eta,0)\,d\eta d\xi\right| \leq MC\left|\iint\limits_{\Omega} |ln\sigma_1\sigma_2|(r_1^2r_2^2)^{-\beta}\,(\xi,\eta)^m d\eta d\xi\right|$$

$$\leq \frac{20}{m+2}MC. \qquad (17)$$

Let n = 1. Then, by condition (5), taking into account (14) and (17), from (15) we obtain

$$|u_2-u_1| \leq \left|\iint\limits_{\Omega} g_{04}(\xi,\eta;x,y\,)f_{1u}(u_1-u_0)(\xi,\eta)^m d\eta d\xi\right| \leq$$

$$\frac{60}{m+2}MCN_1\left|\iint\limits_{\Omega} g_{04}(\xi,\eta;x,y\,)(\xi,\eta)^m d\eta d\xi\right| \leq \frac{20}{m+2}MC\frac{60N_1C}{m+2}.$$

Using the induction method, we obtain

$$|u_{n+1}-u_n| \leq \frac{20}{m+2}MC\left(\frac{60N_1C}{m+2}\right)^n, \mathrm{n} = 2,3 \ldots$$

The Lemma is proved.

From (16) it follows that the series

$$u_0 + \sum_{n=1}^{\infty}(u_n - u_{n-1})$$

uniformly and absolutely converge in $\overline{\boldsymbol{\Omega}}$, if

$$N_1 < (m+2)/(60C).$$

Therefore, the limit:

$$u(x,y) = \lim_{n\to\infty} u_n(x,y)$$

satisfies equation (13), moreover the boundary conditions

$A_s[u]|_{\sigma} = 0,\; 0 < s < l,$

$\mathrm{u(x,y)}|_{\mathrm{OA}} = 0,\; x \in I_1,$

$\mathrm{u(x,y)}|_{\mathrm{OB}} = 0, y \in I_2.$

That was required to prove.

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He graduated from the Tashkent State University. His scientific area are connected with the boundary value problems for the quasilinear equations of the mixed type with two lines of degeneracy. At the present Dr. H.Rasulov is a head of Department of Mathematics at the Bukhara State University.