Volume 6, No.11, November 2019
Journal of Global Research in Mathematical Archives
RESEARCH PAPER
Available online at http://www.jgrma.info
ISSN 2320 – 5822

SOME PROPERTIES OF MIXED FRACTIONAL INTEGRO-DIFFERENTIATION OPERATORS IN HÖLDER SPACES

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Abstract: As is known, the Riemann-Liouville fractional integration operator establishes an isomorphism between Hölder spaces for functions one variables. We study mixed Riemann-Liouville fractional integration operats and mixed fractional derivative in Marchaud form of function of two variables in Hölder spaces of different orders in each variables. The obtained results extend the well known theorem of Hardy-Littlewood for one-dimensuianl fractional integrals to the case of mixed Hölderness.
*Key words***:** functions of two variables, fractional derivative of Marchaud form, mixed fractional derivative, mixed fractional integral, Hölder space.

INTRODUCTION

The classical result of G.Hardi and D.Littlewood (1928, see [1, §3]) is known that the fractional integral $\left(I_{a+}^{\alpha} f\right)(x)=\Gamma^{-1}(\alpha)\left(t_{+}^{\alpha-1} * f\right)(x),\ 0<\alpha<1$ maps isomorphically the space $H_0^{\lambda}([0,1])$ of Hölder order $\lambda\in(0,1)$ functions with a condition $f(0)=0$ on a similar space of a higher order $\lambda+\alpha$ provided that $\lambda+\alpha<1$. Further, this result was generalized in various directions: a space with a power weight, generalized Hölder spaces, spaces of the Nikolsky type, etc. A detailed review of these and some other similar results can be found in [1].

In the multidimensional case, the statement about the properties of a map in Hölder spaces for a mixed fractional Riemann – Liouville integral was studied in [2] - [6].

$$\left(I_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y\frac{\varphi(t,\tau)dtd\tau}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}},\quad x>a,\ y>c, \tag{1}$$

Mixed fractional derivatives form Marchaud ([7]-[9])

$$\left(D_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)=\frac{\varphi(x,y)x^{-\alpha}y^{-\beta}}{\Gamma(1-\alpha)\Gamma(1-\beta)}+\frac{\alpha\beta}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int_0^x\int_0^y\frac{\varphi(x,y)-\varphi(t,\tau)}{(x-t)^{1+\alpha}(y-\tau)^{1+\beta}}dtd\tau, \tag{2}$$

where $x>0,\ y>0$, were not studied either in the usual Hölder space, or in the Hölder spaces defined by mixed differences. Meanwhile, there arise “points of interest” related to the investigation of the above mixed differences of fractional derivatives form Marchaud. For operators (1) in Hölder spaces of mixed order there arise some questions to be answered in relation to the usage of these or those differences in the definition of Hölder spaces. Such mapping properties in Hölder spaces of mixed order were not studied. This paper is aimed to fill in this gap. We deal with non-weighted spaces.

Consider the operator (1) in a rectangle $Q=\{(x,y): 0<x<b,\ 0<y<d\}$.

For a continuous function $\varphi(x,y)$ on R^2 we introduce the notation

$$\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)=\varphi(x+h,y)-\varphi(x,y),\quad \left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)=\varphi(x,y+\eta)-\varphi(x,y),$$

$$\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)=\varphi(x+h,y+\eta)-\varphi(x+h,y)-\varphi(x,y+\eta)+\varphi(x,y),$$

so that

$$\varphi(x+h,y+\eta)=\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)+\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)+\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)+\varphi(x,y). \tag{3}$$

Everywhere in the sequel by C, C_1, C_2 etc we denote positive constants which may different values in different occurences and even in the same line.

Definition 1. Let $\lambda,\gamma\in(0,1]$. We say that $\varphi\in H^{\lambda,\gamma}(Q)$, if

$$|\varphi(x_1,y_1)-\varphi(x_2,y_2)|\le C_1|x_1-x_2|^{\lambda}+C_2|y_1-y_2|^{\gamma} \tag{4}$$

for all $(x_1,y_1),(x_2,y_2)\in Q$. Condition (4) is equivalent to the couple of the separate conditions

$$\left|\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)\right|\le C_1|h|^{\lambda},\qquad \left|\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)\right|\le C_2|\eta|^{\gamma}$$

uniform with respect to another variable.

By $H_0^{\lambda,\gamma}(Q)$ we define a subspace of functions $f\in H_0^{\lambda,\gamma}(Q)$, vanishing at the boundaries $x=0$ and $y=0$ of Q.

Let $\lambda=0$ and /or $\gamma=0$. We put $H^{0,0}(Q)=L^{\infty}(Q)$ and

$$H^{\lambda,0}(Q)=\left\{\varphi\in L^{\infty}(Q):\left|\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)\right|\le C_1|h|^{\lambda}\right\},\ \lambda\in(0,1],$$

$$H^{0,\gamma}(Q)=\left\{\varphi\in L^{\infty}(Q):\left|\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)\right|\le C_2|\eta|^{\gamma}\right\},\ \gamma\in(0,1].$$

Definition 2. We say that $\varphi(x,y)\in\tilde{H}^{\lambda,\gamma}(Q)$, where $\lambda,\gamma\in(0,1]$, if

$$\varphi(x,y)\in H^{\lambda,\gamma}(Q)\ \text{and}\ \left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|\le C_3|h|^{\lambda}|\eta|^{\gamma}. \tag{5}$$

We say that $\varphi(x,y)\in\tilde{H}_0^{\lambda,\gamma}(Q)$, if $\varphi(x,y)\in\tilde{H}^{\lambda,\gamma}(Q)$ and $\varphi(x,y)|_{x=0,\ y=0}=0$.

These spaces become Banach spaces under the standard definition of the

$$\|\varphi\|_{H^{\lambda,\gamma}}:=\|\varphi\|_{C(Q)}+\sup_{x,x+h\in[0,b]}\ \sup_{y\in[0,d]}\frac{\left|\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)\right|}{|h|^{\lambda}}+\sup_{x\in[0,b]}\ \sup_{y,y+\eta\in[0,d]}\frac{\left|\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)\right|}{|\eta|^{\gamma}},$$

$$\|\varphi\|_{\tilde{H}^{\lambda,\gamma}}:=\|\varphi\|_{H^{\lambda,\gamma}}+\sup_{x,x+h\in[0,b]}\ \sup_{y,y+\eta\in[0,d]}\frac{\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|}{|h|^{\lambda}|\eta|^{\gamma}}.$$

Note that

$$\varphi(x,y)\in H^{\lambda,\gamma}\Rightarrow\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|\le C_\theta|h|^{\theta\lambda}|\eta|^{(1-\theta)\gamma},\ \text{for any}\ \theta\in[0,1], \tag{6}$$

where $C_\theta=2C_1^{\theta}C_2^{1-\theta}$, so that

$$\bigcap_{0\le\theta\le1}\tilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q)\hookleftarrow H^{\lambda,\gamma}(Q)\hookleftarrow\tilde{H}^{\lambda,\gamma}(Q) \tag{7}$$

where $\hookleftarrow$ stands for the continuous embedding, and the norm for $\bigcap_{0\le\theta\le1}\tilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q)$ is introduced as the maximum in θ of norms for $\tilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q)$. Since $\theta\in[0,1]$ is arbitarary, it is not hard to see that the inequality in (6) is equivalent to

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|\le C\min\left\{|h|^{\lambda},|\eta|^{\gamma}\right\}. \tag{8}$$

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Theorem 1. Let $\varphi(x,y)\in H^{\lambda,\gamma}(Q)$, $0\le\lambda,\ \gamma\le1$, $0<\alpha,\ \beta<1$. Then for the mixed fractional integral operator (1) the representation

$$\left(I_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)=\frac{\varphi(0,0)}{\Gamma(1+\alpha)\Gamma(1+\beta)}x^{\alpha}y^{\beta}+\frac{y^{\beta}}{\Gamma(1+\beta)}\psi_1(x)+\frac{x^{\alpha}}{\Gamma(1+\alpha)}\psi_2(y)+\psi(x,y) \tag{9}$$

holds, where

$$\psi_1(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{\varphi(t,)-\varphi(0,0)}{(x-t)^{1-\alpha}}dt,\qquad \psi_2(y)=\frac{1}{\Gamma(\beta)}\int_c^y\frac{\varphi(0,s)-\varphi(0,0c)}{(y-s)^{1-\beta}}ds,$$

$$\psi(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y\frac{\left(\overset{1,1}{\Delta}_{t,s}\varphi\right)(0,0)}{(x-t)^{1-\alpha}(y-s)^{1-\beta}}dtds,$$

and

$$|\psi_1(x)|\le C_1x^{\lambda+\alpha},\qquad |\psi_2(y)|\le C_2y^{\gamma+\beta},\tag{10}$$

$$|\psi(x,y)|\le C\min_{0\le\theta\le1}x^{\alpha+\theta\lambda}y^{\beta+(1-\theta)\gamma}=Cx^\alpha y^\beta\min\{x^\lambda,y^\gamma\}.\tag{11}$$

Proof. Representation (9) itself is easily obtained by means of (3). Since $\varphi\in H^{\lambda,\gamma}(Q)$, inequalities (10) are obvious. Estimate (11) is obtained by means of (6) and (8).

Theorem 2. Let $0\le\lambda,\ \gamma\le1$. Then the mixed fractional integral operator $I^{\alpha,\beta}_{0+,0+}$ is bounded from $H_0^{\lambda,\gamma}(Q)$ into $H_0^{\lambda+\alpha,\gamma+\beta}(Q)$, if $\lambda+\alpha<1$ and $\gamma+\beta<1$.

Proof. Since $\varphi(x,y)\in H_0^{\lambda,\gamma}(Q)$, then by (9) we have

$$\left(I^{\alpha,\beta}_{a+,c+}\varphi\right)(x,y)=\psi(x,y).$$

We denote

$$g(x,y)=\left(\overset{1,1}{\Delta}_{x,\,y}\varphi\right)(0,0)\tag{12}$$

for brevity. Note that

$$\left(\overset{1,1}{\Delta}_{x,\,y}\varphi\right)(0,0)=\varphi(x,y)$$

for $\varphi\in H_0^{\lambda,\gamma}$, but we prefer to keep the notetion for $g(x,y)$ via the mixed difference as in (12). By (6) we have

$$|g(x,y)|\le Cx^{\theta\lambda}y^{(1-\theta)\gamma}\le\min\{x^\lambda,\ y^\gamma\}.\tag{13}$$

For $h>0;\ x,\ x+h\in Q_1=[0,b]$, we consider the difference

$$\psi(x+h,y)-\psi(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left(\int_{-h}^x\int_0^y\frac{g(x-t,y-s)}{(t+h)^{1-\alpha}s^{1-\beta}}dtds-\right.$$

$$\left.-\int_0^x\int_0^y\frac{g(x-t,y-s)}{t^{1-\alpha}s^{1-\beta}}dtds\right)=\frac{(x+h)^\alpha-x^\alpha}{\Gamma(\alpha+1)\Gamma(\beta)}\int_0^y\frac{g(x,y-s)}{s^{1-\beta}}ds+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{-h}^0\int_0^y\frac{g(x-t,y-s)-g(x,y-s)}{(t+h)^{1-\alpha}s^{1-\beta}}dtds+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y\frac{g(x-t,y-s)-g(x,y-s)}{s^{1-\beta}}\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]dtds=\Delta_1+\Delta_2+\Delta_3.\tag{14}$$

Since $\forall\theta\in[0,1]$, we make use of (13) with $\theta=1$ and obtain

$$|\Delta_1|\le Cx^\lambda\left|(x+h)^\alpha-x^\alpha\right|\le Ch^{\alpha+\lambda}.$$

For Δ_2 in view of (6), we have

$$|g(x-t,y-s)-g(x,y-s)|=\left|\left(\overset{1,1}{\Delta}_{-t,\,y-s}\varphi\right)(x,0)\right|\le C|t|^\lambda,\tag{15}$$

and then $\Delta_2\le Ch^{\lambda+\alpha}$.

For Δ_3 by (15) and (6) we obtain

$$\Delta_3\le C\int_0^x t^\lambda\left|t^{\alpha-1}-(t+h)^{\alpha-1}\right|dt\le C_0h^{\alpha+\lambda},\quad C_0=\int_0^\infty t^\lambda\left|t^{\alpha-1}-(t+h)^{\alpha-1}\right|dt<\infty.$$

Gathering the estimates $\Delta_1,\Delta_2,\Delta_3$ we obtain

$$|\psi(x+h,y)-\psi(x,y)|\le C_1h^{\lambda+\alpha}.$$

Rearranging symmetrically representation (14), we can similarly obtain that

$$\left|\psi(x, y+\eta)-\psi(x, y)\right| \le C_2 \eta^{\beta+\gamma}.$$

Which proves the theorem.

Theorem 3. The mixed fractional integral operator $I^{\alpha,\beta}_{0+,0+}$ is bounded from the sapce $\widetilde{H}^{\lambda,\gamma}_0(Q)$, $0\le\lambda,\ \gamma\le 1$ into the space $\widetilde{H}^{\lambda+\alpha,\gamma+\beta}_0(Q)$, if $\lambda+\alpha<1$ and $\gamma+\beta<1$.

Proof. Let $\varphi(x, y)\in \widetilde{H}^{\lambda,\gamma}_0(Q)$. By Theorem 2 and embedding (7), for $f(x, y)=\left(I^{\alpha,\beta}_{0+,0+}\varphi\right)(x, y)$ it satisfies to estimate the difference $\left(\overset{1,1}{\Delta}_{h,\eta} f\right)(x, y)$. Since $\varphi(x, y)\big|_{x=0,\ y=0}=0$, according to (9) we have $f(x, y)=\psi(x, y)$, where $\psi(x, y)$ is the function from (9). the main moment in the estimations is to find the corresponding splitting which allows to derive the best information in each variable not losing the corresponding information in another variable. The suggested splitting runs as follows

$$\left(\overset{1,1}{\Delta}_{h,\eta} f\right)(x, y)=\left(\overset{1,1}{\Delta}_{h,\eta} \psi\right)(x, y)=\sum_{k=1}^{9} T_k :=$$

$$:=\frac{g(x, y)\left[(x+h)^{\alpha}-x^{\alpha}\right]}{\Gamma(1+\alpha)\Gamma(1+\beta)}\left[(y+\eta)^{\beta}-y^{\beta}\right]+\frac{(y+\eta)^{\beta}-y^{\beta}}{\Gamma(\alpha)\Gamma(1+\beta)}\int_{-h}^{0}\frac{g(x-t, y)-g(x, y)}{(t+h)^{1-\alpha}}dt+$$

$$+\frac{(x+h)^{\alpha}-x^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta)}\int_{-\eta}^{0}\frac{g(x, y-s)-g(x, y)}{(s+\eta)^{1-\beta}}ds+$$

$$+\frac{(y+\eta)^{\beta}-y^{\beta}}{\Gamma(\alpha)\Gamma(\beta+1)}\int_{0}^{x}\left[g(x-t, y)-g(x, y)\right]\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]dt+$$

$$+\frac{(x+h)^{\alpha}-x^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta)}\int_{0}^{y}\left[g(x, y-s)-g(x, y)\right]\left[(s+\eta)^{\beta-1}-s^{\beta-1}\right]ds+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{-h}^{0}\int_{-\eta}^{0}\frac{\left(\overset{1,1}{\Delta}_{-t,-s} g\right)(x, y)}{(h+t)^{1-\alpha}(s+\eta)^{1-\beta}}dtds+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{-h}^{0}\int_{0}^{y}\frac{\left(\overset{1,1}{\Delta}_{-t,-s} g\right)(x, y)}{(h+t)^{1-\alpha}}\left[(s+\eta)^{\beta-1}-s^{\beta-1}\right]dtds+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{0}^{x}\int_{-\eta}^{0}\frac{\left(\overset{1,1}{\Delta}_{-t,-s} g\right)(x, y)}{(s+\eta)^{1-\beta}}\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]dtds+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{0}^{x}\int_{0}^{y}\left(\overset{1,1}{\Delta}_{-t,-s} g\right)(x, y)\left[(h+t)^{\alpha-1}-t^{\alpha-1}\right]\left[(s+\eta)^{\beta-1}-s^{\beta-1}\right]dtds,$$

where $h,\ \eta>0;\ \ x, x+h\in[0, b],\ \ y, y+\eta\in[0, d]$ and $g(x, y)$ is the function from (12). The validity of this representation may be be chacked directly.

Since $\varphi(x, y)\in \widetilde{H}^{\lambda,\gamma}_0$, we have

$$\left|g(x, y)\right|=\left|\left(\overset{1,1}{\Delta}_{x,\ y}\varphi\right)(0,0)\right|\le Cx^{\lambda}y^{\gamma} \qquad (16)$$

and then

$$\left|T_1\right|\le Cx^{\lambda}\left[(x+h)^{\alpha}-x^{\alpha}\right]y^{\gamma}\left[(y+\eta)^{\beta}-y^{\beta}\right],$$

$$\left|T_2\right|\le Cy^{\gamma}\left[(y+\eta)^{\beta}-y^{\beta}\right]\int_{-h}^{0}\frac{|t|^{\lambda}}{(t+h)^{1-\alpha}}dt,$$

$$|T_3| \le Cx^{\lambda}\left[(x+h)^{\alpha}-x^{\alpha}\right]\int_{-\eta}^{0}\frac{|s|^{\gamma}}{(s+\eta)^{1-\beta}}ds,$$

$$|T_4| \le Cy^{\gamma}\left[(y+\eta)^{\beta}-y^{\beta}\right]\int_{0}^{x} t^{\lambda}\left|(t+h)^{\alpha-1}-t^{\alpha-1}\right|dt,$$

$$|T_5| \le Cx^{\lambda}\left[(x+h)^{\alpha}-x^{\alpha}\right]\int_{0}^{y} s^{\gamma}\left|(s+\eta)^{\beta-1}-s^{\beta-1}\right|ds.$$

For $T_6 - T_9$ we similaryly, make use of

$$\left|\left(\overset{1,1}{\Delta}_{-t,\,-s}\, g\right)(x,y)\right| = \left|\left(\overset{1,1}{\Delta}_{-t,\,-s}\, \varphi\right)(x,y)\right| \le C|t|^{\lambda}|s|^{\gamma}, \tag{17}$$

and obtain

$$|T_6| \le C\int_{-h}^{0}\frac{|t|^{\lambda}}{(h+t)^{1-\alpha}}dt\int_{-\eta}^{0}\frac{|s|^{\gamma}}{(s+\eta)^{1-\beta}}ds,$$

$$|T_7| \le C\int_{-h}^{0}\frac{|t|^{\lambda}}{(h+t)^{1-\alpha}}dt\int_{0}^{y} s^{\gamma}\left|(s+\eta)^{\beta-1}-s^{\beta-1}\right|ds,$$

$$|T_8| \le C\int_{0}^{x} t^{\lambda}\left|(t+h)^{\alpha-1}-t^{\alpha-1}\right|dt\int_{-\eta}^{0}\frac{|s|^{\gamma}}{(s+\eta)^{1-\beta}}ds,$$

$$|T_9| \le C\int_{0}^{x} t^{\lambda}\left|(h+t)^{\alpha-1}-t^{\alpha-1}\right|dt\int_{0}^{y} s^{\gamma}\left|(s+\eta)^{\beta-1}-s^{\beta-1}\right|ds,$$

after which every term is estimated in the standard way, and we get

$$\left|\left(\overset{1,1}{\Delta}_{h,\,\eta}\, f\right)(x,y)\right| \le C_3 h^{\lambda+\alpha}\eta^{\gamma+\beta}. \tag{18}$$

This completes the proof.

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Theorem 4. Let $f(x,y)\in \widetilde{H}^{\lambda,\gamma}(Q),\ \alpha<\lambda\le 1, \beta<\gamma\le 1$. Then for the mixed fractional differential operator (2) the representation

$$\left(D^{\alpha,\beta}_{0+,0+} f\right)(x,y) = \frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\left[\frac{f(0,0)}{x^{\alpha}y^{\beta}}+\frac{\psi_1(x)}{y^{\beta}}+\frac{\psi_2(y)}{x^{\alpha}}+\psi(x,y)\right], \tag{19}$$

and

$$|\psi_1(x)| \le C_1 x^{\lambda-\alpha},\quad |\psi_2(y)| \le C_2 y^{\gamma-\beta}, \tag{20}$$

$$|\psi(x,y)| \le Cx^{\lambda-\alpha}y^{\gamma-\beta} \tag{21}$$

where

$$\psi_1(x) = \frac{f(x,0)-f(0,0)}{x^{\alpha}}+\alpha\int_0^x\frac{f(x,0)-f(t,0)}{(x-t)^{\alpha+1}}dt,$$

$$\psi_2(y) = \frac{f(0,y)-f(0,0)}{y^{\beta}}+\beta\int_0^y\frac{f(0,y)-f(0,\tau)}{(y-\tau)^{1+\beta}}d\tau,$$

$$\psi(x,y) = \frac{1}{x^{\alpha}y^{\beta}}\left(\overset{1,1}{\Delta}_{x,y}\, f\right)(0,0)+\frac{\alpha}{y^{\beta}}\int_0^x\left(\overset{1,1}{\Delta}_{x-t,y}\, f\right)(t,0)\frac{dt}{(x-t)^{1+\alpha}}+$$

$$+\frac{\beta}{x^{\alpha}}\int_0^y\left(\overset{1,1}{\Delta}_{x,y-\tau}\, f\right)(0,\tau)\frac{d\tau}{(y-\tau)^{1+\beta}}+\alpha\beta\int_0^x\int_0^y\left(\overset{1,1}{\Delta}_{x-t,y-\tau}\, f\right)(t,\tau)\frac{dtd\tau}{(x-t)^{1+\alpha}(y-\tau)^{1+\beta}}.$$

Proof. Representation (19) itself is easily obtained by means of (3). Since $f \in H^{\lambda,\gamma}(Q)$, inequalities (2) are obvious. Estimate (21) is obtained by means of (6) and (8).

Theorem 5. Let $f(x,y) \in H_0^{\lambda+\alpha,\gamma+\beta}(Q)$, $\alpha < \lambda \le 1,\ \beta < \gamma \le 1$. Then the operator $\mathbf{D}_{0+,0+}^{\alpha,\beta}$ continuously maps $H_0^{\lambda+\alpha,\gamma+\beta}(Q)$ into $H_0^{\lambda,\gamma}(Q)$.

Proof. Since $f(x,y) \in H_0^{\lambda+\alpha,\gamma+\beta}(Q)$, by (19) we have

$$\varphi(x,y) = \left(\mathrm{D}_{0+,0+}^{\alpha,\beta} f\right)(x,y) = \psi(x,y).$$

By (6) and (8) we have

$$\left|\left(\overset{1,1}{\Delta}_{x,y} f\right)(0,0)\right| \le C x^{\theta(\lambda+\alpha)} y^{(1-\theta)(\gamma+\beta)} \le \min\left\{x^{\lambda+\alpha}, y^{\gamma+\beta}\right\}. \tag{22}$$

Let $h > 0;\ x, x+h \in [0,b]$. We consider the difference

$$\psi(x+h,y) - \psi(x,y) = \frac{\left(\overset{1,1}{\Delta}_{h,y} f\right)(0,0)}{y^\beta (x+h)^\alpha} + \frac{\left(\overset{1,1}{\Delta}_{x,y} f\right)(0,0)}{y^\beta}\left[\frac{1}{(x+h)^\alpha} - \frac{1}{x^\alpha}\right] +$$
$$+ \frac{\alpha}{y^\beta}\int_0^x \frac{\left(\overset{1,1}{\Delta}_{h,y} f\right)(x,0)}{(x+h-t)^{1+\alpha}}dt + \frac{\alpha}{y^\beta}\int_x^{x+h} \frac{\left(\overset{1,1}{\Delta}_{x+h-t,y} f\right)(t,0)}{(x+h-t)^{1+\alpha}}dt + \frac{\beta}{(x+h)^\alpha}\int_0^y \frac{\left(\overset{1,1}{\Delta}_{h,y-\tau} f\right)(0,\tau)}{(y-\tau)^{1+\beta}}d\tau +$$
$$+ \frac{\alpha}{y^\beta}\int_0^x \left(\overset{1,1}{\Delta}_{x-t,y} f\right)(t,0)\left[(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha}\right]dt + \beta\left[(x+h)^{-\alpha} - x^{-\alpha}\right]\int_0^y \frac{\left(\overset{1,1}{\Delta}_{x,y-\tau} f\right)(0,\tau)}{(y-\tau)^{1+\beta}}d\tau +$$
$$+ \alpha\beta\int_0^x\int_0^y \frac{\left(\overset{1,1}{\Delta}_{h,y-\tau} f\right)(x,\tau)dtd\tau}{(x+h-t)^{1+\alpha}(y-\tau)^{1+\beta}} + \alpha\beta\int_x^{x+h}\int_0^y \frac{\left(\overset{1,1}{\Delta}_{x+h-t,y-\tau} f\right)(x,\tau)dtd\tau}{(x+h-t)^{1+\alpha}(y-\tau)^{1+\beta}} +$$
$$+ \alpha\beta\int_0^x\int_0^y \frac{\left(\overset{1,1}{\Delta}_{x-t,y-\tau} f\right)(x,\tau)}{(y-\tau)^{1+\beta}}\left[(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha}\right]dtd\tau\,. \tag{23}$$

We make use of (22) with $\theta = 1$ and obtain

$$|\psi(x+h,y) - \psi(x,y)| \le C_1\left[\frac{h^{\lambda+\alpha}}{(x+h)^\alpha} + x^{\lambda+\alpha}\left[\frac{1}{(x+h)^\alpha} - \frac{1}{x^\alpha}\right] + h^{\lambda+\alpha}\int_0^x \frac{dt}{(x+h-t)^{1+\alpha}} +\right.$$
$$\left. + \int_x^{x+h}\frac{dt}{(x+h-t)^{1-\lambda}} + \int_0^x (x-t)^{\lambda+\alpha}\left[\frac{1}{(x+h-t)^{1+\alpha}} - \frac{1}{(x-t)^{1+\alpha}}\right]dt\right] = \sum_{k=1}^5 \Phi_k\,.$$

For Φ_1 we have

$$\Phi_1 = C_1 h^\lambda \left(\frac{h}{x+h}\right)^\alpha \le Ch^\lambda$$

Let's estimate Φ_2 . Here we shall consider two case: $x \le h$ and $x > h$. In the first case, we use inequality $\left|\sigma_1^\mu - \sigma_2^\mu\right| \le \left|\sigma_1 - \sigma_2\right|^\mu,\ (\sigma_1 \ne \sigma_2)$ and obtain

$$\Phi_2 \le C_1 \frac{x^\lambda h^\alpha}{(x+h)^\alpha} \le Ch^\lambda\,,$$

in second case, using $(1+t)^\alpha - 1 \le \alpha t,\ t > 0$ we have

$$\Phi_2 \le C_1 \frac{x^{\lambda+\alpha-1}h}{(x+h)^\alpha} \le Ch^\lambda\,.$$

For Φ_3 we have:

$$\Phi_3 \le C_1 h^{\lambda+\alpha} \int_0^x \frac{dt}{(t+h)^{1+\alpha}} \le Ch^{\lambda}.$$

For Φ_4

$$\Phi_4 = C_1 \int_x^{x+h} (x+h-t)^{\lambda-1} dt = Ch^{\lambda}.$$

For Φ_5

$$\Phi_5 = C_1 h^{\lambda} \int_0^{x/h} t^{\lambda} \left| \frac{1}{(1+t)^{1+\alpha}} - \frac{1}{t^{1+\alpha}} \right| dt \le Ch^{\lambda}.$$

Gathering the estimates for $\Phi_k,\ k=\overline{1,5}$ we obtaint

$$|\psi(x+h,y)-\psi(x,y)| \le Ch^{\lambda}.$$

Rearranging symmetrically representation (12), we can similarly obtain that

$$|\psi(x,y+\eta)-\psi(x,y)| \le C\eta^{\gamma}.$$

Theorem 6. Let $f(x,y)\in \widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q),\ \alpha<\lambda\le 1,\ \beta<\gamma\le 1$. Then the operator $\mathbf{D}_{0+,0+}^{\alpha,\beta}$ continuously maps $\widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$ into $\widetilde{H}_0^{\lambda,\gamma}(Q)$.

Proof. Let $f(x,y)\in \widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$. Then we have $\varphi(x,y)=\left(\mathrm{D}_{0+,0+}^{\alpha,\beta} f\right)(x,y)=\psi(x,y)$, where $\psi(x,y)$ is the function from (9). The main moment in the estimations is to find the corresponding splitting which allows to derive the best information in each variable not losting the corresponding information in another variable.

Let $h,\eta >;\ x,x+h\in[0,b],\ \ y,y+\eta\in[,\ d]$. We consider the difference

$$\left(\overset{1,1}{\Delta}_{h,\eta}\psi\right)(x,y)=\sum_{k=1}^{25}\Psi_k := \frac{\left(\overset{1,1}{\Delta}_{h,\eta} f\right)(x,y)}{(x+h)^{\alpha}(y+\eta)^{\beta}} + \frac{\left(\overset{1,1}{\Delta}_{h,y} f\right)(x,0)}{(x+h)^{\alpha}}\left[\frac{1}{y^{\beta}}-\frac{1}{(y+\eta)^{\beta}}\right]+$$

$$+\frac{\left(\overset{1,1}{\Delta}_{x,\eta} f\right)(0,y)}{(y+\eta)^{\beta}}\left[\frac{1}{x^{\alpha}}-\frac{1}{(x+h)^{\alpha}}\right]+\left(\overset{1,1}{\Delta}_{xy} f\right)(0,0)\left[\frac{1}{x^{\alpha}}-\frac{1}{(x+h)^{\alpha}}\right]\left[\frac{1}{y^{\beta}}-\frac{1}{(y+\eta)^{\beta}}\right]+$$

$$+\frac{\beta}{(x+h)^{\alpha}}\int_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{h,\ y+\eta-\tau} f\right)(x,\tau)}{(y+\eta-\tau)^{1+\beta}}d\tau+\frac{\beta}{(x+h)^{\alpha}}\int_0^{y}\frac{\left(\overset{1,1}{\Delta}_{h,\ \eta} f\right)(x,y)}{(y+\eta-\tau)^{1+\beta}}d\tau+$$

$$+\beta\left[\frac{1}{x^{\alpha}}-\frac{1}{(x+h)^{\alpha}}\right]\int_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{x,\ y+\eta-\tau} f\right)(0,\tau)}{(y+\eta-\tau)^{1+\beta}}d\tau+$$

$$+\frac{\beta}{(x+h)^{\alpha}}\int_0^{y}\left(\overset{1,1}{\Delta}_{h,\ y-\tau} f\right)(x,\tau)\left[\frac{1}{(y-\tau)^{1+\beta}}-\frac{1}{(y+\eta-\tau)^{1+\beta}}\right]d\tau+$$

$$+\beta\left[\frac{1}{x^{\alpha}}-\frac{1}{(x+h)^{\alpha}}\right]\int_0^{y}\frac{\left(\overset{1,1}{\Delta}_{x,\eta} f\right)(0,y)}{(y+\eta-\tau)^{1+\beta}}d\tau+\frac{\alpha}{(y+\eta)^{\beta}}\int_x^{x+h}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\eta} f\right)(t,y)}{(x+h-t)^{1+\alpha}}dt+$$

$$+\beta\left[\frac{1}{x^{\alpha}}-\frac{1}{(x+h)^{\alpha}}\right]\int_0^{y}\left(\overset{1,1}{\Delta}_{x,y-\tau} f\right)(0,\tau)\left[\frac{1}{y^{1+\beta}}-\frac{1}{(y+\eta)^{1+\beta}}\right]d\tau+$$

$$+\frac{\alpha}{(y+\eta)^{\beta}}\int_0^{x}\frac{\left(\overset{1,1}{\Delta}_{h,\eta} f\right)(x,y)}{(x+h-t)^{1+\alpha}}dt+\alpha\left[\frac{1}{y^{\beta}}-\frac{1}{(y+\eta)^{\beta}}\right]\int_x^{x+h}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,y} f\right)(t,0)}{(x+h-t)^{1+\alpha}}dt+$$

$$+\frac{\alpha}{(y+\eta)^{\beta}}\int_0^x\left(\overset{1,1}{\Delta}_{x-t,\eta} f\right)(t,y)\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dt+$$

$$+\alpha\left[\frac{1}{y^{\beta}}-\frac{1}{(y+\eta)^{\beta}}\right]\int_0^x\frac{\left(\overset{1,1}{\Delta}_{h,y} f\right)(x,0)}{(x+h-t)^{1+\alpha}}dt+$$

$$+\alpha\left[\frac{1}{y^{\beta}}-\frac{1}{(y+\eta)^{\beta}}\right]\int_0^x\left(\overset{1,1}{\Delta}_{x-t,y} f\right)(t,0)\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dt+$$

$$+\int_0^x\int_0^y\frac{\left(\overset{1,1}{\Delta}_{h,\eta} f\right)(x,y)dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}}+\int_0^x\int_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{h,y+\eta-\tau} f\right)(x,\tau)dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}}+$$

$$+\int_0^x\int_0^y\frac{\left(\overset{1,1}{\Delta}_{h,y-\tau} f\right)(x,\tau)}{(x+h-t)^{1+\alpha}}\left[\frac{1}{(y-\tau)^{1+\beta}}-\frac{1}{(y+\eta-\tau)^{1+\beta}}\right]dtd\tau+$$

$$+\int_x^{x+h}\int_0^y\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\eta} f\right)(t,y)dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}}+\int_x^{x+h}\int_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,y+\eta-\tau} f\right)(t,\tau)dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}}+$$

$$+\int_x^{x+h}\int_0^y\frac{\left(\overset{1,1}{\Delta}_{x+h-t,y-\tau} f\right)(t,\tau)}{(x+h-t)^{1+\alpha}}\left[\frac{1}{(y-\tau)^{1+\beta}}-\frac{1}{(y+\eta-\tau)^{1+\beta}}\right]dtd\tau+$$

$$+\int_0^x\int_0^y\frac{\left(\overset{1,1}{\Delta}_{x-t,\eta} f\right)(t,y)}{(y+\eta-\tau)^{1+\beta}}\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{x-t,y+\eta-\tau} f\right)(t,\tau)}{(y+\eta-\tau)^{1+\beta}}\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dtd\tau+$$

$$+\int_0^x\int_0^y\left(\overset{1,1}{\Delta}_{x-t,y-\tau} f\right)(t,\tau)\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]\left[\frac{1}{(y-\tau)^{1+\beta}}-\frac{1}{(y+\eta-\tau)^{1+\beta}}\right]dtd\tau\,.$$

The validity of this representation may be checked directly. Since $f(x,y)\in\widetilde{H}_0^{\lambda,\gamma}(Q)$, we have

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}\psi\right)(x,y)\right|\le\sum_{k=1}^{25}\left|\Psi_k\right|\le C\left[\frac{h^{\lambda}\eta^{\gamma}}{(x+h)^{\alpha}(y+\eta)^{\beta}}+\frac{h^{\lambda}y^{\gamma}}{(x+h)^{\alpha}}\left[\frac{1}{y^{\beta}}-\frac{1}{(y+\eta)^{\beta}}\right]+\right.$$

$$+\frac{x^{\lambda}\eta^{\gamma}}{(y+\eta)^{\beta}}\left[\frac{1}{x^{\alpha}}-\frac{1}{(x+h)^{\alpha}}\right]+x^{\lambda}y^{\gamma}\left[\frac{1}{x^{\alpha}}-\frac{1}{(x+h)^{\alpha}}\right]\left[\frac{1}{y^{\beta}}-\frac{1}{(y+\eta)^{\beta}}\right]+$$

$$+\frac{h^{\lambda}}{(x+h)^{\alpha}}\int_y^{y+\eta}(y+\eta-\tau)^{\gamma-1-\beta}d\tau+\frac{\beta h^{\lambda}\eta^{\gamma}}{(x+h)^{\alpha}}\int_0^y\frac{d\tau}{(y+\eta-\tau)^{1+\beta}}++\beta x^{\lambda}\left[\frac{1}{x^{\alpha}}-\frac{1}{(x+h)^{\alpha}}\right]\int_y^{y+\eta}(y+\eta-\tau)^{\gamma-1-\beta}d\tau+$$

$$+\frac{h^{\lambda}\beta}{(x+h)^{\alpha}}\int_0^y(y-\tau)^{\gamma}\left[\frac{1}{(y-\tau)^{1+\beta}}-\frac{1}{(y+\eta-\tau)^{1+\beta}}\right]d\tau+$$

$$+x^{\lambda}\eta^{\gamma}\beta\left[\frac{1}{x^{\alpha}}-\frac{1}{(x+h)^{\alpha}}\right]\int_0^y\frac{d\tau}{(y+\eta-\tau)^{1+\beta}}+\frac{\eta^{\gamma}\alpha}{(y+\eta)^{\beta}}\int_x^{x+h}(x+h-t)^{\lambda-1-\alpha}dt+$$

$$+x^{\lambda}\beta\left[\frac{1}{x^{\alpha}}-\frac{1}{(x+h)^{\alpha}}\right]\int_0^y(y-\tau)^{\gamma}\left[\frac{1}{y^{1+\beta}}-\frac{1}{(y+\eta)^{1+\beta}}\right]d\tau+$$

$$+\frac{\alpha h^{\lambda}\eta^{\gamma}}{(y+\eta)^{\beta}}\int_0^x\frac{dt}{(x+h-t)^{1+\alpha}}+\alpha y^{\gamma}\left[\frac{1}{y^{\beta}}-\frac{1}{(y+\eta)^{\beta}}\right]\int_x^{x+h}(x+h-t)^{\lambda-1-\alpha}dt+$$

$$+\frac{\alpha\eta^{\gamma}}{(y+\eta)^{\beta}}\int_0^x(x-t)^{\lambda}\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dt+$$

$$+\alpha h^{\lambda}y^{\gamma}\left[\frac{1}{y^{\beta}}-\frac{1}{(y+\eta)^{\beta}}\right]\int_0^x\frac{dt}{(x+h-t)^{1+\alpha}}+$$

$$+\alpha y^{\gamma}\left[\frac{1}{y^{\beta}}-\frac{1}{(y+\eta)^{\beta}}\right]\int_0^x(x-t)^{\lambda}\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dt+$$

$$+h^{\lambda}\eta^{\gamma}\int_0^x\int_0^y\frac{dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}}+\int_0^x\int_y^{y+\eta}\frac{h^{\lambda}dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta-\gamma}}+$$

$$+\int_0^x\int_0^y\frac{h^{\lambda}(y-\tau)^{\gamma}}{(x+h-t)^{1+\alpha}}\left[\frac{1}{(y-\tau)^{1+\beta}}-\frac{1}{(y+\eta-\tau)^{1+\beta}}\right]dtd\tau+$$

$$+\int_x^{x+h}\int_0^y\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\eta}f\right)(t,y)dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}}+\int_x^{x+h}\int_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,y+\eta-\tau}f\right)(t,\tau)dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}}+$$

$$+\int_x^{x+h}\int_0^y\frac{(y-\tau)^{\gamma}}{(x+h-t)^{1+\alpha-\lambda}}\left[\frac{1}{(y-\tau)^{1+\beta}}-\frac{1}{(y+\eta-\tau)^{1+\beta}}\right]dtd\tau+$$

$$+\int_0^x\int_0^y\frac{(x-t)^{\lambda}\eta^{\gamma}}{(y+\eta-\tau)^{1+\beta}}\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta}\frac{(x-t)^{\lambda}}{(y+\eta-\tau)^{1+\beta-\gamma}}\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dtd\tau+$$

$$+\int_0^x\int_0^y(x-t)^{\lambda}(y-\tau)^{\gamma}\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]\left[\frac{1}{(y-\tau)^{1+\beta}}-\frac{1}{(y+\eta-\tau)^{1+\beta}}\right]dtd\tau\Bigg].$$

After which every term is estimated in the standard way, and we get

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|\le C_3h^{\lambda}\eta^{\gamma}.$$

This completes the proof.

Main theorem. The mixed fractional integral operator $I^{\alpha,\beta}_{0+,0+}$ isomorphically maps the space $\widetilde{H}^{\lambda,\gamma}_0(Q),\ 0\le\lambda,\gamma\le1$ onto the space $\widetilde{H}^{\lambda+\alpha,\gamma+\beta}_0(Q)$, if $\lambda+\alpha<1$ and $\gamma+\beta<1$.

Proof. We should consider, as usual the following three parts of the proof:

1) Action of the mixed fractional integral operator from the space $\widetilde{H}^{\lambda,\gamma}_0(Q)$ to the space $\widetilde{H}^{\lambda+\alpha,\gamma+\beta}_0(Q)$;
2) Action of the mixed fractional differentiation operator from the space $\widetilde{H}^{\lambda+\alpha,\gamma+\beta}_0(Q)$ to the space $\widetilde{H}^{\lambda,\gamma}_0(Q)$;
3) The possibility to represent any function $f(x,y)\in\widetilde{H}^{\lambda+\alpha,\gamma+\beta}_0(Q)$ as $\left(I^{\alpha,\beta}_{0+,+}\varphi\right)(x,y)$ with the density in $\widetilde{H}^{\lambda,\gamma}_0(Q)$.

Because of (1) the parts 1) -2) are covered by Theorems 3 and 6. The part 3) is treated in the standart way in case $0<\alpha<1$ and $0<\beta<1$ by using the possibility of similar representation with the density from $L_{\bar p}(R^2)$, $\bar p=(p_1,p_2)$. See [1] Theorem 24.4.

REFERENCES

[1]. Samko S.G., A.A. Kilbas and O.I. Marichev. Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach. Sci. Publ., N. York - London, 1993, 1012 pp. (Russian Ed. - Fractional Interals and Derivatives and Some of Their Applications, Nauka i Tekhnika, Minsk, 1987).

[2]. Mamatov T. Mixed fractional integration operators in Hölder spaces (Russian). «Science and World». Volgograd, № 1 (1). 2013, P. 30-38

[3]. Mamatov T, S. Samko. Mixed Fractional Integration Operators in Mixed Weighted Hölder Spaces. FC&AA. Vol.13, Num 3. 2010, p. 245-259

[4]. Mamatov T. Weighted Zygmund estimates for mixed fractional integration. Case Studies Journal. Volume 7, Issue 5, 2018. p. 82-88

[5]. Mamatov T. Mixed Fractional Integration In Mixed Weighted Generalized Hölder Spaces. Case Studies Journal. Volume 7, Issue 6. 2018, P. 61-68

[6]. Mamatov T. Mixed Fractional Integration Operators in Mixed Weighted Hölder Spaces. Monograph. LAPLAMBERT Academic Publishing. P. 73

[7]. Mamatov T. Mixed Fractional Integro-Differentiation Operators in Hölder Spaces. The latest research in modren science: experience, traditions and innovations. Proceedings of the VII International Scientific Conference. Section I. North Charleston, SC, USA. 20-21 June, 2018. P. 6-9

[8]. Mamatov T., Rayimov D., Elmurodov M. Mixed Fractioanl Differentiation Operators in Hölder Spaces. Journal of Multidisciplinary Engineering Science and Technology (JMEST), Vol. 6 Issue 4, April – 2019. P. 9855-9857

[9]. Mamatov T. Mapping properties of mixed fractional differentiation operators in Hölder spaces defined by usual Hölder condition. Journal of Computer Science & Computational Mathematics, Volume 9, Issue 2, June 2019. DOI: 10.20967/jcscm.2019.02.003

